CE 311K - McKinney

HW-10 Curve Fitting: Regression and Interpolation

1. Given the data

x	5	10	15	20	25	30	35	40	45	50
y	16	25	32	33	38	36	39	40	42	42

Use least-squares regression to fit the following equations to the data in the table above (Compare your results by preparing a plot of the data and each of your equations on a single graph using Excel).

(a) a straight line: $y_i = a_0 + a_1 x_i$

(b) a parabola: $y_i = a_0 + a_1 x_i + a_2 x_i^2$

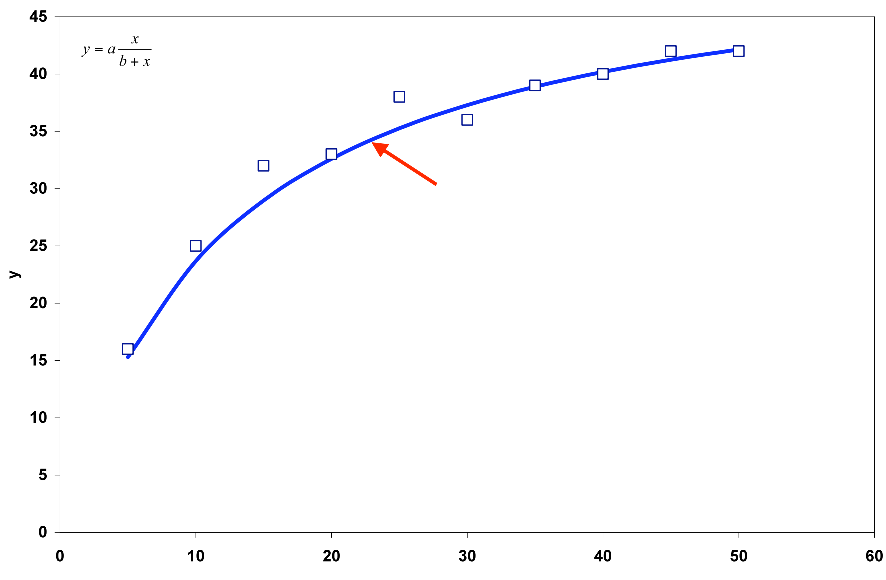
(c) a power equation: $y = ax^b$; and

(c) a saturation-growth-rate equation: $y = a\dfrac{x}{b+x}$ (SEE NOTE BELOW)

NOTE: A saturation-growth-rate equation is a nonlinear model that is sometimes fitted to data. The form of the equation is

$$y = a\frac{x}{b+x}$$

where a and b are constant coefficients. For example, the following graph shows data and a saturation – growth curve from the equation above.



This equation is often used to model population growth under conditions where the population y levels off (saturates) as x increases. To linearize the equation we can invert it to give

$$\frac{1}{y} = \frac{b}{a}\frac{1}{x} + \frac{1}{a}$$

Now, the equation can be written in terms of the new variables $Y = 1/y$ and $X = 1/x$

$$Y = mX + s$$

Thus a plot of Y versus X is a straight line, with a slope of $m = b/a$ and an intercept of $s = 1/a$.

(a) Straight line $y = a_0 + a_1 x$

	x	y	x^2	xy
	5	16	25	80
	10	25	100	250
	15	32	225	480
	20	33	400	660
	25	38	625	950
	30	36	900	1080
	35	39	1225	1365
	40	40	1600	1600
	45	42	2025	1890
	50	42	2500	2100
Sums	275	343	9625	10455

$$a_0 = \frac{\frac{1}{10}(343)(9625) - \frac{1}{10}(275)(10455)}{(9625) - \frac{1}{10}[275]^2} = 20.667$$

$$a_1 = \frac{(10455) - \frac{1}{10}(275)(343)}{9625 - \frac{1}{10}[275]^2} = 0.4958$$

so $y = 20.667 + 0.4958x$

(b) Parabola $y = a_0 + a_1 x + a_2 x^2$

	x	y	x^2	X^3	X^4	xy	x^2y
	5	16	25	125	625	80	400
	10	25	100	1000	10000	250	2500

	15	32	225	3375	50625	480	7200
	20	33	400	8000	160000	660	13200
	25	38	625	15625	390625	950	23750
	30	36	900	27000	810000	1080	32400
	35	39	1225	42875	1500625	1365	47775
	40	40	1600	64000	2560000	1600	64000
	45	42	2025	91125	4100625	1890	85050
	50	42	2500	125000	6250000	2100	105000
Sums	275	343	9625	378125	15833125	10455	381275

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} x_i^2 y_i \end{bmatrix}$$

$$\begin{bmatrix} 10 & 275 & 9625 \\ 275 & 9625 & 378125 \\ 9625 & 378125 & 15833125 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 343 \\ 10455 \\ 381275 \end{bmatrix}$$

$$a_0 = 12.167$$
$$a_1 = 1.3458$$
$$a_2 = -0.0155$$

$$y = 12.167 + 1.13458x - 0.0155x^2$$

(c) Power equation $y = ax^b$ or $\ln y = \ln a + b \ln x$

	X=lnx	Y=lny	X^2	XY
	1.6094	2.7726	2.5903	4.4623
	2.3026	3.2189	5.3019	7.4117
	2.7081	3.4657	7.3335	9.3854
	2.9957	3.4965	8.9744	10.4746
	3.2189	3.6376	10.3612	11.7089
	3.4012	3.5835	11.5681	12.1883
	3.5553	3.6636	12.6405	13.0252
	3.6889	3.6889	13.6078	13.6078
	3.8067	3.7377	14.4907	14.2280
	3.9120	3.7377	15.3039	14.6218
Sums	31.1988	35.0026	102.1724	111.1142

$$a_0 = \frac{\frac{1}{10}(35.002)(102.17) - \frac{1}{10}(31.199)(111.11)}{(102.17) - \frac{1}{10}\left[31.199\right]^2} = 2.2678$$

$$a_1 = \frac{(111.11) - \frac{1}{10}(31.199)(35.002)}{(102.17) - \frac{1}{10}\left[31.199\right]^2} = 0.39503$$

But since $a_0 = \ln a$ and $a_1 = b$, we have

$$a = e^{a_0} = e^{2.2678} = 9.658 \qquad b = 0.39503$$

Therefore

$$y = ax^b = 9.658x^{0.39505}$$

(d) Saturation-growth-rate equation $\frac{1}{y} = \frac{b}{a}\frac{1}{x} + \frac{1}{a}$

	X=1/x	Y=1/y	X^2	XY
	0.2000	0.0625	0.0400	0.0125
	0.1000	0.0400	0.0100	0.0040
	0.0667	0.0313	0.0044	0.0021
	0.0500	0.0303	0.0025	0.0015
	0.0400	0.0263	0.0016	0.0011
	0.0333	0.0278	0.0011	0.0009
	0.0286	0.0256	0.0008	0.0007
	0.0250	0.0250	0.0006	0.0006
	0.0222	0.0238	0.0005	0.0005
	0.0200	0.0238	0.0004	0.0005
Sums	0.5858	0.3164	0.0620	0.0244

$$a_0 = \frac{\frac{1}{10}(3164)(0.0620) - \frac{1}{10}(0.5858)(0.0244)}{(0.0620) - \frac{1}{10}\left[0.5858\right]^2} = 0.019142$$

$$a_1 = \frac{(0.0244) - \frac{1}{10}(0.5858)(0.3164)}{(0.0620) - \frac{1}{10}\left[0.5858\right]^2} = 0.21337$$

But since $a_0 = 1/a$ and $a_1 = b/a$, we have

$$a = 5.2241 \qquad b = 11.147$$

Therefore

$$y = a\frac{x}{b + x} = 5.2241\frac{x}{11.147 + x}$$