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NUMERICAL DIFFUSION AND DISSIPATION IN HYDROSTATIC MODELS OF INTERNAL WAVES

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ABSTRACT

Analysis of numerical diffusion and dissipation rates in a hydrostatic model of an internal wave are used to illustrate convergence problems associated with hydrostatic modeling of an essentially non-hydrostatic phenomenon. As the model grid is refined in the vertical direction, the numerical diffusion is convergent (i.e. reduces with grid refinement). However as the model grid is refined in the horizontal direction, the diffusive error increases with grid refinement. The error behavior is linked to the development of shorter wavelength horizontal waves that are numerically generated as the internal wave steepens. Higher grid refinement in the horizontal direction allows shorter wavelengths to be developed, which have higher rates of numerical diffusion and dissipation. It is seen that the onset of the shorter wavelengths coincides with a shift in the diffusion and dissipation behavior that is highly grid-dependent.

Keywords: internal wave, hydrodynamic numerical model, numerical error

INTRODUCTION

The behavior of internal waves in lakes and coastal oceans may have an impact on mixing across the thermocline, sediment resuspension as well as the distribution of phytoplankton and nutrients. Due to practical computational limits, models used for lake and coastal ocean systems are typically based on the Reynolds-averaged Navier-Stokes equations with the hydrostatic approximation. As the hydrostatic approximation requires larger horizontal velocity scales than vertical – a condition readily met over much of such systems – the approximation appears reasonable. However, it is a fundamental tenet of numerical modeling that a good model must be convergent as the grid scale is refined. This brings up a difficulty for models applying the hydrostatic approximation: solutions on refined grids may converge to the wrong equations! That is, as the model grid is refined we reduce numerical errors in dissipation and diffusion, arriving at a better *hydrostatic* solution for scales where *non-hydrostatic* behavior may be

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important. We can consider model "skill" to be how well a model represents the physical phenomena, from which it follows that "skill" considers both the errors introduced in numerical approximation and the errors introduced by approximation in the governing equations. Thus, a hydrostatic model on a refined grid may not be more skillful than the model on a coarse grid as reduction in numerical error may increase the accumulation of error associated with the hydrostatic approximation. This appears to have a significant impact on accurate modeling of internal waves.

Internal waves are affected both by nonlinear steepening and dispersion, the latter being inherently non-hydrostatic and therefore neglected in a hydrostatic model. The evolution of non-breaking internal waves from basin-scale seiches into trains of solitary waves was analyzed and demonstrated in the laboratory by Horn et al. (2001). As hydrostatic models retain the nonlinear terms that steepen an internal wave, there is a tendency for waves in a hydrostatic model to steepen until controlled by dissipation or diffusion (either modeled or numerical). When numerical error controls steepening, the effect will be greater at finer grid scales. The increased steepening can lead to a net reduction of model skill with grid refinement. This effect was first noted as an aside in a conference paper by Laval et al. (2000), but led to much consternation among the reviewers of the subsequent journal paper (Laval et al. 2003) and was excised as being too controversial and parenthetical to the main thrust of the research. The present paper provides a more detailed look at the error behavior associated with grid refinement of internal waves in hydrostatic models.

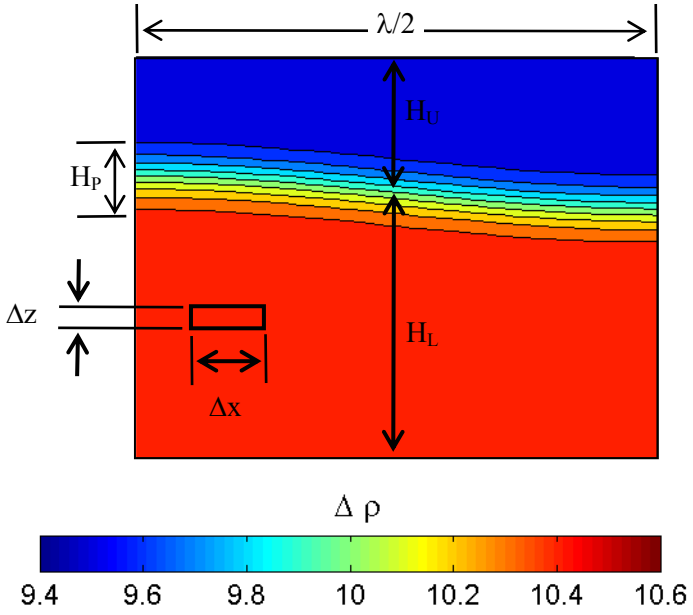

FIG. 1. Definitional sketch. The following characteristics were used for all simulations: H_P = 15 m, H_L = 32.92 m, H_U = 16.96 m, λ = 2x10^4 m, initial wave amplitude = 2 m. Reference density = 1000 kg/m^3

RESULTS AND DISCUSSION

To illustrate the error increase associated with grid refinement in a model of an internal wave, we can consider a simple inviscid two-dimensional domain (Fig. 1). The initial condition is a horizontal-mode-one, basin-scale cosine wave with the density profile obtained from a hyperbolic tangent. The result is a pycnocline (H_P) with an almost linear density profile that smoothly transitions between two regions of essentially constant density. For a system with zero viscosity and zero scalar diffusivity, the theoretical wave evolution provides strict conservation of energy. Furthermore, while kinetic and available potential energy should smoothly oscillate with the wave motion, the background potential energy (i.e. the lowest potential energy possible for an adiabatically resorted density field, e.g. Winters et al., 1995) must remain constant. Thus, by applying a model with zero physical viscosity and zero physical diffusivity, we can use the energy changes to quantify the errors associated with numerical dissipation and diffusion as the grid is refined.

The model used in this study is the Centre for Water Research Estuary and Lake Computer Model (CWR-ELCOM), which is based on established methodologies (see Hodges, 2000, for details) and has seven papers on methods and results that are published or in press. CWR-ELCOM solves the three-dimensional unsteady RANS equations using: 1) either 1st or 2nd order semi-implicit free surface solution, 2) quadratic semi-Lagrangian advection discretization and 3) third-order ULTIMATE QUICKEST (Leonard, 1991) scalar transport.

We have conducted a series of model runs at varying grid resolutions illustrated in TABLE 1. Set A refines the grid in the vertical, while holding the horizontal grid fixed. Set B refines the grid in the horizontal while holding the vertical grid fixed. The time step was held constant with 150 time steps per wave period ($T/\Delta t = 150$).

TABLE 1. Model configurations

	Set A	Set B
horizontal grid resolution ($\lambda_0 / \Delta x$)	60	30 60 120
vertical grid resolution ($H_P / \Delta z$)	6 9 12	12

To analyze the system at each time step, we compute the background density as the density field if the internal wave were brought to rest without mass diffusion. The background potential energy (E_B) is the potential energy of this background density field. For an internal wave system with zero mass diffusion, E_B should be constant for all time – i.e. pure advection cannot cause diffusion, and only diffusion can change E_B. Thus, the increase in background potential energy is a direct measure of the accumulated effects of numerical diffusion for this model setup.

The available potential energy (E_A) is the difference between the potential energy (E_P) at a time step and the background, ie.

$$E_A = E_P - E_B \tag{1}$$

As an internal wave creates an oscillation between E_A and kinetic energy (E_K), we can define the

"dynamic" energy (E_D) of the system as their sum:

$$E_D = E_A + E_K \tag{2}$$

For an inviscid internal wave system, E_D should be constant for all time. Thus, numerical dissipation can be analyzed from the decline in the dynamic energy.

For Set A, refining the vertical grid, the results in Fig 2(a) show that the differences in dynamic energy are small up until ~ 7 wave periods, when the dissipation rate for the more refined grid becomes slightly greater than that for the coarser grids. In Fig 2(b), it is seen that the more highly resolved grid always shows a smaller increase in the background potential energy. Thus, it appears that up until 7 wave periods, refining the vertical grid behaves as we would like a model to behave, i.e. refining the grid reduces both the dissipative and diffusive errors.

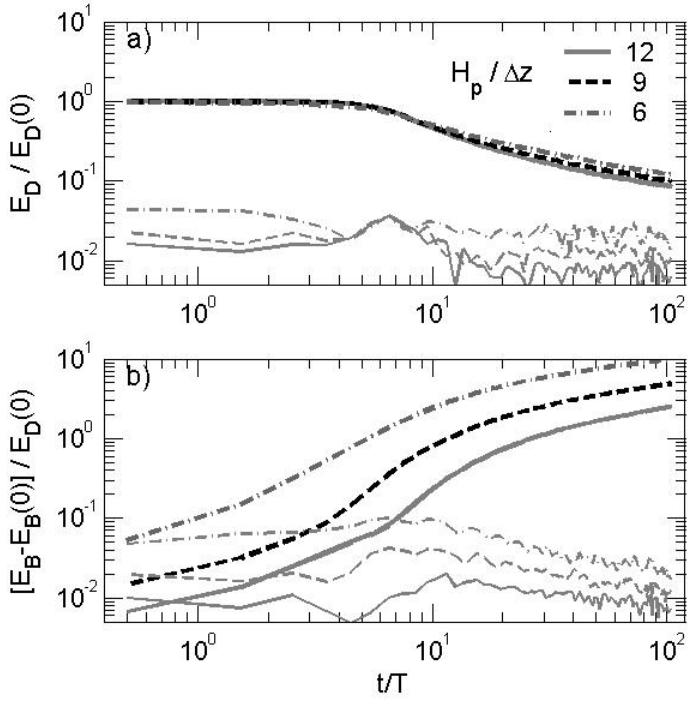

FIG. 2. Refining vertical grid resolution (Set A). Thick lines are time evolution of a) dynamic energy and b) background potential energy. Vertical grid resolution is defined by the thickness of the pycnocline (H_p) normalized by the vertical grid cell thickness (Δz); Dynamic energy, E_D, is normalized by the initial dynamic energy, $E_D(0)$; Background potential energy, E_B, is presented as increase from the initial background potential energy, $E_B(0)$, normalized by the initial dynamic energy. Model results are mean values for 1 wave-period bins. Thin lines at bottom of each panel provide standard deviation of bins.

For Set B, refining the horizontal grid, the results in Fig 3(a) are similar to Fig. 2(a), showing only small differences in dissipation up until ~ 7 wave periods, where the dissipation rate increases and the more refined grids have greater dissipation. Fig. 3(b) is dramatically different than Fig. 2(b) in that it shows that horizontal grid refinement leads to substantially greater diffusive error after ~ 7 wave periods.

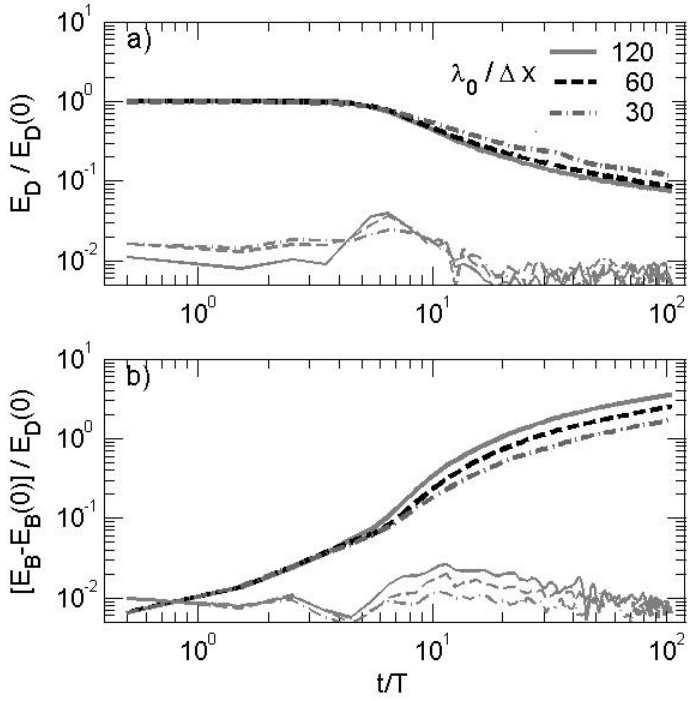

FIG. 3. Refining horizontal grid resolution (Set B). Thick lines are time evolution of a) dynamic energy and b) background potential energy. Horizontal grid resolution is defined by the initial condition wavelength (λ_0) normalized by the horizontal grid cell length (Δx); Other normalization and lines as in Fig 2.

To understand why the characteristics of the error change around 7 wave periods, we can examine the power spectral density (PSD) of the internal wave displacement at each time step. In Fig 4, the evolution of the PSD is presented in terms of the normalized wavelength, so that at the start of the simulation all the power is concentrated in a wave whose normalized wavelength is unity. There can be seen a distinct transition around t/T=7, when the energy substantially transitions into shorter wavelengths. The nonlinear terms allow the wave to steepen and, without any dynamic pressure to limit steepening, the wave front becomes a propagating bore and begins to transfer energy to shorter wavelengths. In both the coarse grid (Fig 4a) and the fine grid (Fig 4b) it can be seen that energy is initially transferred to a short wavelength. As the simulation progresses, the peak energy appears in longer and longer wavelengths (converging

back on unity) so that the figures appear to show the energy cascading up to larger scales. However, this is an artifact of the normalization method used to emphasize the energy at the smaller wavelengths. The PSD in Fig 4 is of the normalized wave displacement for each wave period. Since the wave displacement grows smaller with time (due to numerical dissipation), the structure of the PSD is difficult to decipher unless normalized. Since the overall energy is decaying with time, the apparent transfer of energy from small to larger wavelengths can be better understood as increased dissipation of the energy at the smaller wavelengths, which results in the peak energy shifting back towards the larger scale.

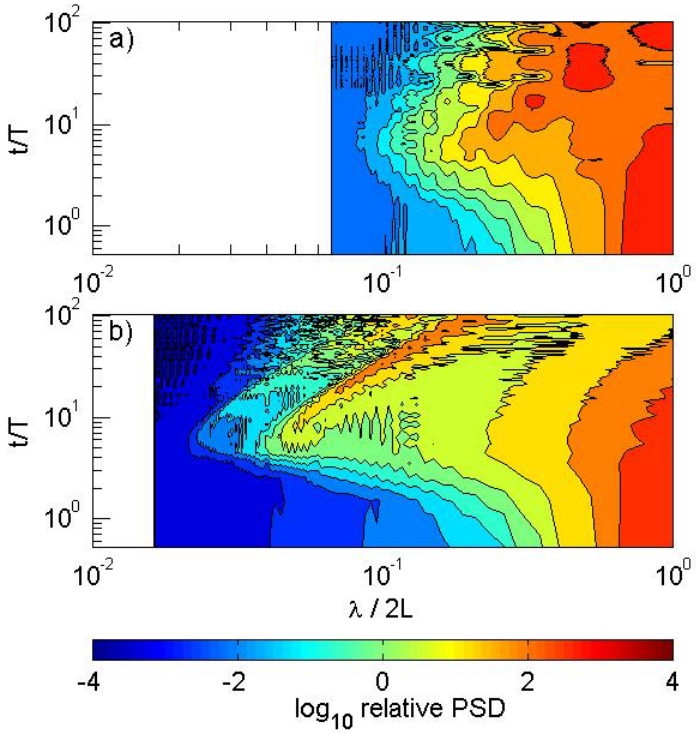

FIG. 4. Wave-period averaged time evolution of relative power spectral density (PSD) computed from internal wave displacement normalized by maximum displacement in each wave period. Panel a) shows PSD for low horizontal grid resolution ($\lambda_0 / \Delta x = 30$). Panel b) shows PSD for high horizontal grid resolution ($\lambda_0 / \Delta x = 120$).

CONCLUSIONS

While the above results do not exhaustively analyze the possible data space associated with internal wave modeling, they are indicative of a fundamental problem with hydrostatic modeling of internal waves. As the model grid is refined in the horizontal direction, the steepening of the wave due to the lack of non-hydrostatic pressure results in energy transfer to smaller wavelengths where numerical diffusion and dissipation are enhanced. If the vertical grid spacing is refined

while holding the horizontal grid spacing constant, then a model is enforcing a greater "hydrostatic" aspect ratio on the computation and the model diffusive error is convergent. However, when the horizontal grid is refined while holding the vertical grid constant, the diffusive error is not convergent, but increases with grid refinement.

ACKNOWLEDGMENTS

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APPENDIX I. NOTATION

λ	wavelength
$\Delta\rho$	local density anomaly
Δx	grid spacing in horizontal
Δz	grid spacing in vertical
E_A	available potential energy
E_B	background potential energy
E_D	dynamic energy
E_K	kinetic energy
E_P	total potential energy
H_L	thickness of the lower density layer
H_P	thickness of pycnocline
H_U	thickness of the upper density layer
L	basin horizontal length
T	horizontal mode-one theoretical internal wave period
t	time