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A SECOND-ORDER CORRECTION FOR SEMI-IMPLICIT SHALLOW WATER METHODS

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ABSTRACT

Test simulations of a free-surface seiche are used to examine the effects of a second-order correction term in a semi-implicit hydrostatic Navier-Stokes solver. Previously the correction term has been neglected, which formally reduces a Crank-Nicolson solution method to first order. However, for the test cases examined to date, the effect of neglecting the correction appears to be small, particularly where the flow depth is significantly larger than the free surface wave amplitude.

Keywords: hydrostatic, Navier-Stokes equations, free surface, shallow water equations

INTRODUCTION

Semi-implicit methods for solving the shallow water equations are commonly used in modeling the hydrodynamics of coastal oceans, lakes, and estuaries. The free surface evolution is modeled with an equation derived by integrating the continuity equation over the depth and applying the kinematic boundary condition at the free surface. Finite-difference solution methods for the equation set are typically represented as first-order or second-order accurate in time, based on using either backwards Euler or Crank-Nicolson discretization for the free surface evolution equation. However, the typical derivation of the semi-implicit method approximates the system depth at time 'n+1' by the depth at time 'n' in the integral momentum term (e.g. Casulli and Cattani, 1994; Hodges, 2000). This approach neglects an implicit nonlinear coupling between the free surface and velocity, allowing the free surface evolution to be predicted by a linear algebraic equation. This approximation introduces an error that is formally first-order in time. A correction term that is formally second-order accurate has been developed and tested to examine the difference between solutions with and without the correction.

DISCUSSION

A free surface evolution equation is derived by integrating incompressible continuity from the bottom boundary (b) to the free surface (h) and applying the kinematic free surface boundary condition. For a 2D free surface in Cartesian coordinates this provides

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$$\frac{\partial h}{\partial t} = - \frac{\partial}{\partial x}\int_b^h u\,dz - \frac{\partial}{\partial y}\int_b^h v\,dz \tag{1}$$

Applying Crank-Nicolson (C-N) discretization results in a temporally second-order accurate discrete form for the free surface evolution. For simplicity, we will write only the 1D form here,

$$h^{n+1} = h^n - \frac{\Delta t}{2}\frac{\delta}{\delta x}\left\{\int_b^{h^n} u^n dz^n + \int_b^{h^{n+1}} u^{n+1} dz^{n+1}\right\} + O\left(\Delta t^3\right) \tag{2}$$

Previously, the occurrence of h^{n+1} in the second integral limit on the R.H.S. has been neglected, which results in an error term of the form

$$\frac{\Delta t}{2}\frac{\delta}{\delta x}\left\{\int_{h^n}^{h^{n+1}} u^{n+1} dz^{n+1}\right\} \tag{3}$$

It can be shown that this term is formally $O(\Delta t^2)$, which invalidates the 2nd order accuracy of the C-N method used in the discretization. However, it is reasonable to hypothesize that this term may not be significant for many simulations, since it is a small portion of the overall integral term on the R.H.S. of eq (2).

To determine the effect of this term on the order of accuracy of a solution, the Center for Water Research Estuary and Lake Computer Model, CWR-ELCOM (Hodges, 2000; Hodges et al., 2000), was modified from a first-order, backwards-Euler, free-surface solution to a C-N solution that could be run with and without a discrete version of the correction term in eq (3). The modeled evolution of an initially-linear 2D seiche in a simple rectangular domain was examined and the error computed to determine model accuracy. The domain length was 10989 m. The water surface was initialized with a simple cosine wave of amplitude 0.1 m. The water depth was successively set as 10.35 m, 3.45 m and 1.35 m to examine effects of nonlinearity associated with different amplitude/depth (a/D) ratios, which were approximately 0.01, 0.03 and 0.1, respectively.

A "converged" solution for making error comparisons was modeled for each a/D ratio, using a time step approximately 1/5000th of the wave period – this is more than an order of magnitude smaller than the smallest time step used in the test cases. As the test cases are inherently unsteady, there is not a true "converged" solution for computing the error, so it is necessary to evaluate the error over a selected interval of the simulation. The error accumulates with time in an oscillating closed-basin simulation, so the error values depend, to some extent, upon the time interval selected; however, the accuracy results show similar behavior when computed over any consistent interval that is sufficiently short that the rate of error accumulation can be neglected. Each test case was run for 20 wave periods, with the surface shape between periods 17 and 19 used to compute the error. Error was computed by taking the root-mean-square (RMS) of the difference between a test solution and the converged solution across the entire free surface at each time step, and then taking the RMS of the time step results over the selected wave periods. The RMS error was normalized by the RMS value of the initial surface wave deflection. Thus, an error of 1.0 would occur with a free surface that was perfectly flat, while an error of 2.0 would be a wave exactly out of phase. Error results are shown in Figure 1 for a grid resolution of 300 m x 0.3 m.

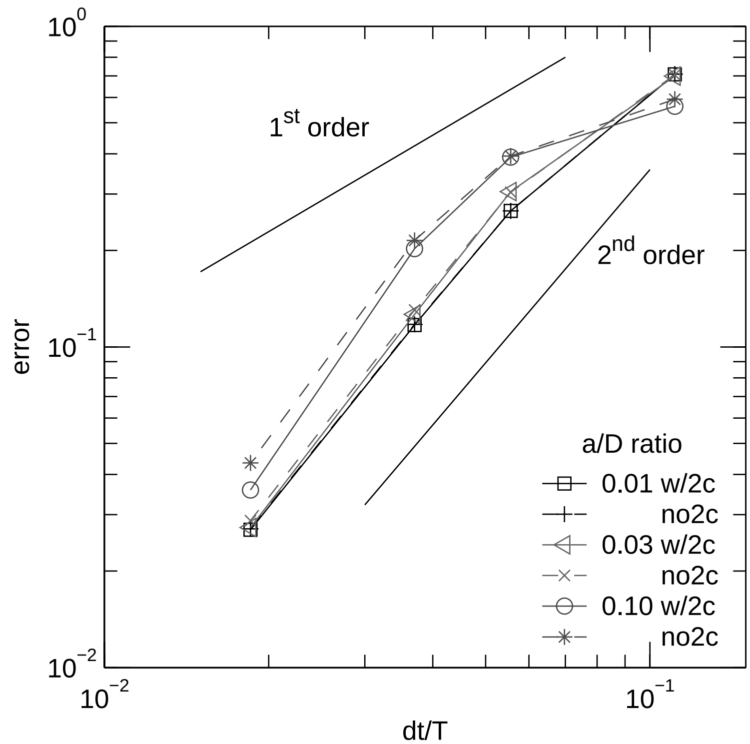

FIG. 1. RMS error in free surface position between wave periods 17 and 19, normalized by RMS of the initial free surface displacement, and shown as a function of the time step (dt) normalized by the wave period (T). Solid lines (w/2D) are results applying the second order correction term, while dashed lines (no2c) do not have the correction term.

CONCLUSIONS

Initial results shown in Figure 1 indicate a very slight improvement in the model accuracy with the addition of the second-order correction (w/2c) as compared to the code without the term (no2c). It is noted that as the a/D ratio increases, the nonlinearity of the free-surface increases and the effect of the second-order correction is more significant. In total, the work to date appears to indicate that the second-order correction is not necessary for most practical applications of hydrostatic models. As a caveat, the above analysis combines both amplitude and phase error in the same measure. Preliminary analysis of phase errors indicate that the 2nd order correction may have a more significant affect on the accuracy of the wave phase, which may imply the above results are only valid when amplitude damping is the dominant error. Thus, the relationship between numerical dissipation and dispersion, which is strongly affected by discretization of the momentum equation, may play a role in determining whether the correction term is required to achieve acceptable model skill.

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