Recirculation and the equilibrium displacement of the thermocline in a wind-driven stratifed lake

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1. Abstract

The equilibrium response of a lake to an applied wind stress has previously been approximated using a two-dimensional, two-density-layer, closed basin with a simple static balance between the barotropic and baroclinic tilts. The next level of sophistication is to allow the wind stress to be distributed over some reduced depth of the upper layer, giving rise to downwind transport near the surface and upwind transport above the thermocline. The existence of the recirculation amplifies or reduces the equilibrium response of the thermocline depending upon the ratio of the wind-mixed layer depth to the epilimnion depth. A deep thermocline (relative to the wind-mixed layer depth) will have a reduced response to wind forcing solely due to the existence of a recirculating flow. These results are of particular interest in the design of numerical models for stratified lakes as they highlight the critical importance of the turbulence model used to capture the wind-mixed layer dynamics: to get the setup of basin-scale internal waves modeled correctly in a stratified lake, it is necessary to have a turbulence model that accurately captures the wind-mixed layer depth.

2. Previous work

The first-order equilibrium response of a stratified lake to an applied wind stress can be approximated from a hydrostatic balance between wind-introduced momentum and the barotropic and baroclinic displacements (Spigel and Imberger, 1980). For a two-layer basin, the hydrostatic equilibrium slope for the internal interface (η_2) is

$$\frac{\partial \eta_2}{\partial x} \approx -\frac{u_*^2}{g' h_1} \tag{1}$$

where u_* is the surface friction velocity due to wind stress, h_1 is the depth of the upper layer (epilimnion), and $g' = g\Delta\rho/\rho$ is the reduced gravity due to stratification. Eq. (1) implies the wind momentum is introduced uniformly over the upper layer and exactly balances the barotropic tilt (which is also in balance with the the baroclinic tilt). This result neglects the dynamic effects of wind-driven circulation. In Heaps and Ramsbottom (1966), the effects of recirculation on lake motion were included in an analytical approach by assuming that a two-layer system could be approximated using constant and uniform eddy viscosities in each layer. This approach was extended and amplified in Heaps (1984), resulting in the equilibrium state of the interface approximated as

$$\frac{\partial \eta_2}{\partial x} \approx -\frac{u_*^2}{g' h_1}\left\{1 + \frac{2}{3}\left(1+\delta_2\right)\lambda + \frac{\lambda h_1}{h_2}\right\} \tag{2}$$

where h_2 is the thickness of the lower layer (hypolimnion) and δ_2 and λ are constant functions of density stratification, layer heights, eddy viscosity and boundary conditions. Where the bottom boundary condition is free slip, this simplifies to

$$\frac{\partial \eta_2}{\partial x} \approx -\frac{u_*^2}{g' h_1}\left\{1 + \lambda\frac{h_1 + h_2}{h_2}\right\} \tag{3}$$

where $0 \leq \lambda < 0.5$. In the above formulation, the coefficient multiplying $u_*^2\left(g' h_1\right)^{-1}$ is always larger than unity, implying recirculation always *increases* the equilibrium tilt. Of course, the derivation of Heaps (1984)

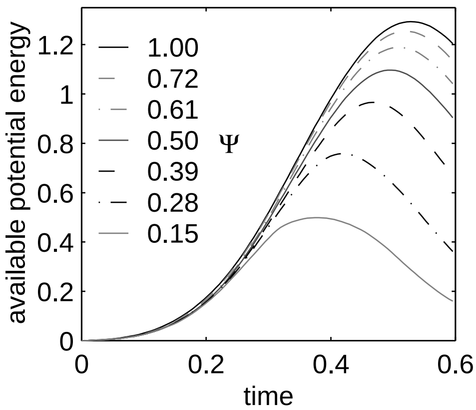

Figure 1: Available potential energy (PE_a) evolution over time. Results are from numerical simulations for two-layer seiche in a two-dimensional rectangular basin. The individual lines represent different simulations where the ratio of the wind-mixed layer depth to the epilimnion depth (Ψ) is varied from 0.15 to 1.0. Ordinate (time) begins at onset of wind with the thermocline at rest and is normalized by the theoretical period of the first-horizontal-mode, first-vertical-mode internal seiche (T_i). Coordinate is simulation PE_a normalized by the theoretical PE_a at maximum thermocline displacement (occurring at $T_i/2$).

is strictly valid only for constant eddy-viscosity, so it should not be surprising if this does not hold true with more complex turbulence closures. Indeed, in developing a turbulence model suitable for stratified lakes (Hodges et al., 2000), it was noted that the form of the turbulence model could significantly alter the setup of internal waves. In particular, it was found that a reduced depth of the wind-mixed layer would *reduce* the available potential energy introduced into the seiching thermocline, as shown in Fig. 1.

3. Interface tilt with recirculation

In an attempt to explain numerical results that are at odds with the approach of Heaps (1984), we can model a narrow lake as a system that has two density layers (h_1, h_2) and three momentum layers (h_d, h_u and h_2, as shown in Fig. 2). Performing a simple force balance at steady state, the work done by the wind in the

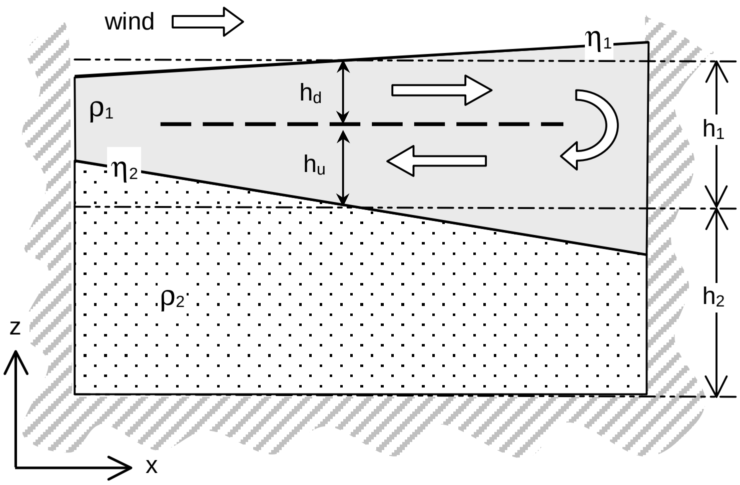

Figure 2: Two-density-layer system with recirculation in the upper density layer (h_1) separated into downwind (h_d) and upwind (h_u) layers.

downwind layer ($u_*^2 h_d^{-1}$) must be balanced by the pressure work of the barotropic tilt and energy dissipation. If we assume the losses are proportional to the square of the mean downwind velocity (represented by the flow rate $Q = u_d h_d$) then we have

$$\frac{u_*^2}{h_d} = g\frac{\partial \eta_1}{\partial x} + \alpha \frac{Q^2}{h_d^2} \tag{4}$$

where α is the unknown dissipation scale factor. In the upwind layer, the balance is between the driving barotropic pressure force and the loss term

$$g\frac{\partial \eta_1}{\partial x} = \alpha \frac{Q^2}{h_u^2} \tag{5}$$

where we have made the tenuous assumption that the scale factor α is equal in both downwind and upwind layers. We also assume the upwind flow rate is equal to the downwind flow rate: a requirement for 2D flows that may not be satisfied in a 3D domain. Substituting Eq. (5) into (4) and solving for α provides

$$\alpha = \left(\frac{h_d^2\, h_u^2}{h_d^2 + h_u^2} \right) \frac{u_*^2}{Q^2\, h_d} \tag{6}$$

If we neglect the effects of recirculation and dissipation in the hypolimnion, the barotropic and baroclinic tilts must balance:

$$g'\frac{\partial \eta_2}{\partial x} = -g\frac{\partial \eta_1}{\partial x} \tag{7}$$

Substituting Eq. (6) and (7) into (4) obtains a representation of the baroclinic tilt as

$$\frac{\partial \eta_2}{\partial x} = -\frac{u_*^2}{g'\, h_1}\left(1 + \frac{h_u}{h_d} - \frac{h_u^2}{h_u^2 + h_d^2} - \frac{h_u^3}{h_d^3 + h_u^2 h_d} \right) \tag{8}$$

This can be written as a factor β multiplying the static equilibrium setup of Eq. (1)

$$\frac{\partial \eta_2}{\partial x} = -\beta \frac{u_*^2}{g'\, h_1} \tag{9}$$

Let $\Psi \equiv h_d\ (h_u + h_d)^{-1}$, then β is

$$\beta = 1 - \frac{\left(1 - 3\Psi + 2\Psi^2\right)}{\left(1 - 2\Psi + 2\Psi^2\right)} \tag{10}$$

The theoretical relationship between the equilibrium tilt amplification (β) and the relative wind-mixed layer depth (Ψ) is shown in Fig. 3, along with typical values of β produced in several numerical simulations with different *a priori* limitations on Ψ. It is clear that our simple separation of the epilimnion into downwind and upwind layers with dissipation related by a scale factor α is only qualitative in its representation of the simulation behaviour. However, the theory does provide evidence that reduction of available potential energy as a function of mixed-layer depth as seen in simulations (Fig. 1 *cf.*) is a result of the recirculation dynamics rather than a numerical aberration. As both the simulation and the theoretical β are unity where the wind-mixed-layer depth is exactly half the epilimnion depth, we can infer the approximation of a uniform α for both layers is reasonable for this special case. As the thicknesses of the downwind and upwind layers diverge from this special case, it may be speculated that the dissipation should include a term that scales on the velocity shear. It is also likely that the effects of wind stirring provide additional dissipation in the downwind surface layer that would not occur in the upwind layer. Finally, the additional dissipation of energy at the end walls has not been included in the theory, providing a further reason for the divergence of results.

This simple theory for equilibrium displacement shows more severe consequences for underpredicting the mixed-layer depth than is seen in the simulations. However, the available potential energy of the internal seiche is a function of the square of the tilt (and therefore of β^2), so the important point remains that moderate underprediction of the mixed layer depth in a numerical simulation may result in a significant underprediction of energy in the internal seiche, as illustrated in Fig. 1. This has critical implications for

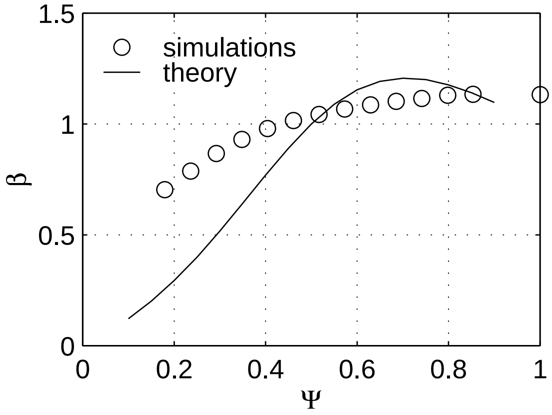

Figure 3: Relationship between the equilibrium displacement multiplier (β) and the ratio of the wind-mixed-layer depth to epilimnion depth (Ψ). Each circle represents a separate simulation where the wind-mixed layer depth is explicitly controlled in the turbulence model.

numerical models of stratified lakes as the potential energy stored in the seiching thermocline is the primary source of energy for advective transport in the hypolimnion.

Acknowledgments

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