Accuracy Order of Crank–Nicolson Discretization for Hydrostatic Free-Surface Flow

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Abstract: Application of Crank–Nicolson (CN) discretization to the hydrostatic (or shallow-water) free-surface equation in two-dimensional or three-dimensional Reynolds-averaged Navier–Stokes models neglects a second order term. The neglected term is zero at steady state, so it does not appear in steady-state accuracy analyses. A new correction term is derived that restores second-order accuracy. The correction is significant when the amplitude of the surface oscillation is within two orders of magnitude of the water depth and the barotropic Courant–Friedrichs–Lewy (CFL) stability condition is less than unity. Analysis shows that the CN accuracy for an unforced free-surface oscillation is degraded to first order when the barotropic CFL stability condition is greater than unity, independent of whether or not the new correction term is applied. The results indicate that the semi-implicit Crank–Nicolson method, applied to the hydrostatic free-surface evolution equation, is only first-order accurate for the time and space scales typically used in lake, estuarine, and coastal ocean studies.

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Introduction

This work was motivated by observations that did not match the underlying accuracy order of the numerical algorithm for semi-implicit solutions of the three-dimensional (3D) hydrostatic Navier–Stokes equations applied to unforced oscillations of a free surface. In the course of investigation, it was determined that (1) the standard approach to second-order temporal discretization of the semi-implicit method neglects a term that is formally second order, (2) irrespective of the neglect of this second-order term, the Crank–Nicolson (CN) method is only first-order accurate for the free surface evolution when the barotropic Courant–Friedrichs–Lewy (CFL) stability condition is greater than unity, and (3) the theta method may be less than first-order accurate for a barotropic CFL stability condition greater than 0.5.

Three-dimensional simulations of lakes, estuaries, and coastal oceans are typically predicated upon solution of the hydrostatic Reynolds-averaged Navier–Stokes (RANS) equations formulated in primitive variables (velocity, water surface height, and density). As barotropic motions in such systems are typically an order of magnitude faster than advective or baroclinic motions, the numerical stability of a discrete algorithm is generally controlled by the free surface discretization. Efficient finite-difference/volume numerical methods use a split form of the governing equations with either explicit or semi-implicit time discretization of free surface gravity waves. For explicit methods (e.g., Blumberg and Mellor 1987), the barotropic mode is solved at a fine time step, providing the effective hydrostatic pressure gradient for a coarse advective/baroclinic time step. For semi-implicit methods (e.g., Casulli and Cheng 1992; Ahsan and Blumberg 1999), a predictor–corrector approach employs an explicit velocity advance as the source for an implicit solution of the free surface evolution, which is used to correct the velocity. The semi-implicit approach allows a single time step to be used for coupling barotropic, baroclinic, and advective motions, while providing a stable solution even when the barotropic CFL stability condition is significantly greater than unity. Both explicit and semi-implicit splitting methods have proven successful at providing efficient and stable solutions of the hydrostatic equations of motion.

Semi-implicit finite difference schemes for coastal, estuarine, and inland water applications have typically been formulated with first or second-order temporal accuracy (backwards Euler, CN, leapfrog, theta method; e.g., Casulli and Cattani 1994). In this paper, a series of test cases are examined to illustrate the temporal order of accuracy of a common semi-implicit method (CN). It is shown that the typical approach to second-order discretization actually neglects a second-order term, so the order of accuracy is formally degraded to first-order in time. However, the effect of this error is small for many practical discretization scales as it arises from an unsteady temporal term and may be dominated by spatial error. More importantly, it is shown that both first- and second-order semi-implicit methods have degraded accuracy order when the barotropic CFL stability condition is greater than unity. That is, the very condition that motivates the semi-implicit method (stable solution at a high barotropic CFL stability condition) leads to model accuracy that may be one order lower than theory.

The present work builds on prior derivations of a first-order semi-implicit method (Casulli and Cheng 1992), and the theta discretization applied to the same model (Casulli and Cattani 1994). The derivations in these prior works cannot be extended

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