Elementary Mechanics of Fluids

Flow in Pipes
Reynolds Experiment

- Reynolds Number

- Laminar flow: Fluid moves in smooth streamlines
- Turbulent flow: Violent mixing, fluid velocity at a point varies randomly with time
- Transition to turbulence in a 2 in. pipe is at $V=2$ ft/s, so most pipe flows are turbulent

\[
\text{Re} = \frac{\rho V D}{\mu}
\]

- Laminar flow \( h_f \propto V \)
- Transition flow \( 2000 \leq \text{Re} < 4000 \)
- Turbulent flow \( h_f \propto V^2 \)

![Diagram of laminar and turbulent flow](image.png)

Laminar \hspace{1cm} \text{Turbulent}
Shear Stress in Pipes

- Steady, uniform flow in a pipe: momentum flux is zero and pressure distribution across pipe is hydrostatic, equilibrium exists between pressure, gravity and shear forces

\[
\sum F_s = 0 = pA - (p + \frac{dp}{ds}\Delta s)A - \Delta W \sin \alpha - \tau_0(\pi D)\Delta s
\]

\[
0 = \frac{dp}{ds}\Delta s A - \gamma A \Delta s \frac{dz}{ds} - \tau_0(\pi D)\Delta s
\]

\[
\tau_0 = \frac{D}{4} \left[-\frac{d}{ds} \gamma (\frac{p}{\gamma} + z)\right]
\]

\[
\tau_0 = -\frac{D \gamma dh}{4 \frac{ds}{dA}}
\]

\[
h_1 - h_2 = h_f = \frac{4L \tau_0}{\gamma D}
\]

- Since \( h \) is constant across the cross-section of the pipe (hydrostatic), and \(-dh/ds > 0\), then the shear stress will be zero at the center (\( r = 0 \)) and increase linearly to a maximum at the wall.

- Head loss is due to the shear stress.

- Applicable to either laminar or turbulent flow

- Now we need a relationship for the shear stress in terms of the Re and pipe roughness
Darcy-Weisbach Equation

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\tau_0 & \rho & V & \mu & D & e \\
\hline
ML^{-1}T^{-2} & ML^{-1} & LT^{-1} & ML^{-1}T^{-1} & L & L \\
\hline
\end{array}
\]

\[\tau_0 = F(\rho, V, \mu, D, e)\]
\[\pi_4 = F(\pi_1, \pi_2)\]
Repeating variables: \(\rho, V, D\)
\[\pi_1 = \text{Re}; \quad \pi_2 = \frac{e}{D}; \quad \pi_3 = \frac{\tau_0}{\rho V^2}\]
\[\frac{\tau_0}{\rho V^2} = F(\text{Re}, \frac{e}{D})\]
\[\tau_0 = \rho V^2 F(\text{Re}, \frac{e}{D})\]

\[h_f = \frac{4L}{\gamma D} \tau_0\]
\[= \frac{4L}{\gamma D} \rho V^2 F(\text{Re}, \frac{e}{D})\]
\[= \frac{L V^2}{D 2g} \left[8F(\text{Re}, \frac{e}{D})\right]\]
\[h_f = f \frac{L V^2}{D 2g}\]
\[f = 8F(\text{Re}, \frac{e}{D})\]

Darcy-Weisbach Eq. Friction factor
Laminar Flow in Pipes

- Laminar flow -- Newton’s law of viscosity is valid:

\[
\tau = \mu \frac{dV}{dy} = -\frac{r \gamma dh}{2 ds}
\]

\[
\frac{dV}{dy} = \frac{-dV}{dr}
\]

\[
\frac{dV}{dr} = \frac{r \gamma dh}{2 \mu ds}
\]

\[
dV = \frac{r \gamma dh dr}{2 \mu ds}
\]

\[
V = \frac{r^2 \gamma dh}{4 \mu ds} + C
\]

\[
C = -\frac{r_0^2 \gamma dh}{4 \mu ds}
\]

\[
V = \frac{-r_0^2 \gamma dh}{4 \mu ds} \left[ 1 - \left( \frac{r}{r_0} \right)^2 \right]
\]

\[
V = V_{max} \left[ 1 - \left( \frac{r}{r_0} \right)^2 \right]
\]

- Velocity distribution in a pipe (laminar flow) is parabolic with maximum at center.
Discharge in Laminar Flow

\[ V = -\frac{\gamma}{4\mu} \frac{dh}{ds} (r_0^2 - r^2) \]

\[ Q = \int V \, dA = \int_0^0 -\frac{\gamma}{4\mu} \frac{dh}{ds} (r_0^2 - r^2) (2\pi r dr) \]

\[ = \frac{\pi \gamma}{4\mu} \frac{dh}{ds} \left( \frac{r^2 - r_0^2}{2} \right) \bigg|_0^0 \]

\[ Q = -\frac{\pi \gamma r_0^4}{8\mu} \frac{dh}{ds} \]

\[ = -\frac{\pi \gamma D^4}{128\mu} \frac{dh}{ds} \]

\[ \bar{V} = \frac{Q}{A} \]

\[ \bar{V} = -\frac{\gamma D^2}{32\mu} \frac{dh}{ds} \]
Head Loss in Laminar Flow

\[
\bar{V} = -\frac{\gamma b^2}{32 \mu} \frac{dh}{ds}
\]

\[
dh = -\bar{V} \frac{32 \mu}{\gamma b^2} ds
\]

\[
dh = -\bar{V} \frac{32 \mu}{\gamma b^2} ds
\]

\[
h_2 - h_1 = -\bar{V} \frac{32 \mu}{\gamma b^2} (s_2 - s_1)
\]

\[
h_1 = h_2 + h_f
\]

\[
h_f = \frac{32 \mu L \bar{V}}{\gamma b^2}
\]

\[
h_f = \frac{32 \mu L \bar{V}}{\gamma b^2} \frac{\rho \bar{V}^2}{2}
\]

\[
h_f = 64 \left( \frac{\mu}{\rho \bar{V} b} \right) \frac{L}{D} \frac{\rho \bar{V}^2}{2}
\]

\[
h_f = 64 \left( \frac{L}{D} \right) \frac{\rho \bar{V}^2}{2}
\]

\[
h_f = f \frac{L}{D} \frac{\rho \bar{V}^2}{2} \quad f = \frac{64}{\text{Re}}
\]
Nikuradse’s Experiments

- In general, friction factor
  \[ f = F(\text{Re}, \frac{e}{D}) \]
  - Function of \textit{Re} and \textit{roughness}
- Laminar region
  \[ f = \frac{64}{\text{Re}} \]
  - Independent of roughness
- Turbulent region
  - Smooth pipe curve
    - All curves coincide @ ~\text{Re}=2300
  - Rough pipe zone
    - All rough pipe curves flatten out and become independent of Re
  \[ f = \frac{0.25}{\log_{10} \left( \frac{e}{3.7D} + \frac{5.74}{\text{Re}^{0.9}} \right)^2} \]
Moody Diagram

\[ \text{Re}^{1/2} = \frac{D^{3/2}}{v} \left( \frac{2gh}{L} \right)^{1/2} \]
Pipe Entrance

- Developing flow
  - Includes boundary layer and core,
  - Viscous effects grow inward from the wall
- Fully developed flow
  - Shape of velocity profile is same at all points along pipe

\[
\frac{L_e}{D} \approx \begin{cases} 
0.06 \text{Re} & \text{Laminar flow} \\
4.4\text{Re}^{1/6} & \text{Turbulent flow}
\end{cases}
\]
Entrance Loss in a Pipe

• In addition to frictional losses, there are minor losses due to
  – Entrances or exits
  – Expansions or contractions
  – Bends, elbows, tees, and other fittings
  – Valves

• Losses generally determined by experiment and then correlated with pipe flow characteristics

• Loss coefficients are generally given as the ratio of head loss to velocity head

\[ K = \frac{h_L}{\frac{V^2}{2g}} \quad \text{or} \quad h_L = K \frac{V^2}{2g} \]

• \( K \) – loss coefficient
  – \( K \approx 0.1 \) for well-rounded inlet (high \( Re \))
  – \( K \approx 1.0 \) abrupt pipe outlet
  – \( K \approx 0.5 \) abrupt pipe inlet

Abrupt inlet, \( K \approx 0.5 \)
Elbow Loss in a Pipe

- A piping system may have many minor losses which are all correlated to $V^2/2g$
- Sum them up to a total system loss for pipes of the same diameter

$$h_L = h_f + \sum h_m = \frac{V^2}{2g} \left[ \frac{f}{D} L + \sum K_m \right]$$

- Where,

  $h_L$ = Total head loss
  $h_f$ = Frictional head loss
  $h_m$ = Minor head loss for fitting $m$
  $K_m$ = Minor head loss coefficient for fitting $m$
EGL & HGL for Losses in a Pipe

- Entrances, bends, and other flow transitions cause the EGL to drop an amount equal to the head loss produced by the transition.
- EGL is steeper at entrance than it is downstream of there where the slope is equal the frictional head loss in the pipe.
- The HGL also drops sharply downstream of an entrance.
Ex(10.2)

**Given:** Liquid in pipe has $\gamma = 8 \text{kN/m}^3$. Acceleration = 0.

$D = 1 \text{ cm}$, $\mu = 3 \times 10^{-3} \text{ N-m/s}^2$.

**Find:** Is fluid stationary, moving up, or moving down?

What is the mean velocity?

**Solution:** Energy eq. from $z = 0$ to $z = 10 \text{ m}$

$$\alpha_1 \frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 - h_L = \alpha_2 \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2$$

$$\frac{200,000}{8000} - h_L = \frac{110,000}{8000} + 10$$

$h_L = \frac{90}{8} - 10$

$h_L = 1.25 \text{ m} \text{ (moving upward)}$

$$h_L = \frac{32\mu LV}{\gamma D^2}$$

$$V = h_L \frac{\gamma D^2}{32\mu L}$$

$$V = 1.25 \frac{8000*(0.01)^2}{32*3\times10^{-3}*10}$$

$V = 1.04 \text{ m/s}$
Ex (10.4)

- Given: Oil \((S = 0.97, \mu = 10^{-2} \text{ lbf-s/ft}^2)\) in a 2 in pipe, \(Q = 0.25 \text{ cfs}\).
- Find: Pressure drop per 100 ft of horizontal pipe.
- Solution:

\[
V = \frac{Q}{A} = \frac{0.25}{\pi (2/12)^2 / 4} = 11.46 \text{ ft/s}
\]

\[
\text{Re} = \frac{\rho V D}{\mu} = \frac{0.97 \times 1 / 94 \times 11.46 \times (2/12)}{10^{-2}} = 360 \text{ (laminar)}
\]

\[
\Delta p = \frac{32 \mu L V}{D^2} = \frac{32 \times 10^{-2} \times 100 \times 11.46}{(2/12)^2} = 91.7 \text{ psi/100 ft}
\]
Ex. (10.8)

Given: Kerosene \((S=0.94, \mu=0.048 \text{ N-s/m}^2)\). Horizontal 5-cm pipe. \(Q=2\times10^{-3} \text{ m}^3/\text{s}\).

Find: Pressure drop per 10 m of pipe.

Solution:

\[
\frac{a_1}{2g} \frac{V_1^2}{\gamma} + p_1 + z_1 = \frac{a_2}{2g} \frac{V_2^2}{\gamma} + p_2 + z_2
\]

\[
h_L = \frac{32\mu L V}{\gamma D^2}
\]

\[
0 + 0 + 0.5 - \frac{32\mu L V}{\gamma D^2} = \frac{a_2}{2g} \frac{V_2^2}{\gamma} + 0 + 0
\]

\[
\frac{a_2}{2g} \frac{V_2^2}{\gamma} + \frac{32\mu L}{\gamma D^2} V - 0.5 = 0
\]

\[
\frac{2}{2g} \frac{V_2^2}{\gamma} + \frac{32 \times 4 \times 10^{-5} \times 10}{0.8 \times 62.4 \times (1/32)^2} V - 0.5 = 0
\]

\[
V_2^2 + 8.45V - 16.1 = 0
\]

\[
V = 1.60 \text{ ft/s}
\]

\[
Re = \frac{0.8 \times 1.94 \times 1.6 \times (0.25/12)}{4 \times 10^{-5}} = 1293 \text{ (laminar)}
\]

\[
Q = V \times A = 1.6 \times \pi \times (0.25/12)^2 / 4 = 1.23 \times 10^{-3} \text{ cfs}
\]
Ex. (10.34)

Given: Glycerin at 20°C flows commercial steel pipe.
Find: Δh
Solution: \( \gamma = 12,300 \, N/m, \mu = 0.62 \, Ns/m^2 \)

\[
\alpha_1 \frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 - h_L = \alpha_2 \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2
\]

\[
\frac{p_1}{\gamma} + z_1 - h_L = \frac{p_2}{\gamma} + z_2
\]

\[\Delta h = \frac{p_1}{\gamma} + z_1 - \left( \frac{p_2}{\gamma} + z_2 \right) = h_L\]

\[\text{Re} = \frac{\rho V D}{\mu} = \frac{V D}{\nu} = \frac{0.6 \times 0.02}{5.1 \times 10^{-4}} = 23.5 \text{ (laminar)}\]

\[\Delta h = h_L = \frac{32 \mu L V}{\gamma D^2} = \frac{32(0.62)(1)(0.6)}{12,300 \times (0.02)^2} = 2.42 \, m\]
Ex. (10.43)

Given: Figure
Find: Estimate the elevation required in the upper reservoir to produce a water discharge of 10 cfs in the system. What is the minimum pressure in the pipeline and what is the pressure there?

Solution:

\[
\alpha_1 \frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 - \sum h_L = \alpha_2 \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2
\]

\[
0 + 0 + z_1 - \sum h_L = 0 + 0 + z_2
\]

\[
\sum h_L = \left( K_e + 2 K_b + K_E + f \frac{L}{D} \right) \frac{V^2}{2g}
\]

\[
K_e = 0.5; \ K_b = 0.4 \text{(assumed)}; \ K_E = 1.0; \ f \frac{L}{D} = 0.025 \times \frac{430}{1} = 10.75
\]

\[
V = \frac{Q}{A} = \frac{10}{\pi/4 \times 1^2} = 12.73 \text{ ft/s}
\]

\[
z_1 = 100 + (0.5 + 2 \times 0.4 + 1.0 + 10.75) \frac{12.73^2}{2 \times 32.2} = 133 \text{ ft}
\]

\[
\alpha_1 \frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 - \sum h_L = \alpha_b \frac{V_b^2}{2g} + \frac{p_b + z_b}{\gamma}
\]

\[
0 + 0 + z_1 - \sum h_L = 1 \times \frac{V_b^2}{2g} + \frac{p_b + z_b}{\gamma}
\]

\[
\frac{p_b}{\gamma} = z_1 - z_b - \frac{V_b^2}{2g} - \left( K_e + K_b + f \frac{L}{D} \right) \frac{V^2}{2g}
\]

\[
= 133 - 110.7 - \left( 1.0 + 0.5 + 0.4 + 0.025 \frac{300}{1} \right) \frac{12.73^2}{2 \times 32.2}
\]

\[
= -1.35 \text{ ft}
\]

\[
p_b = 62.4 \times (-1.53) = -95.9 \text{ psig}
\]

\[
Re = \frac{VD}{\nu} = \frac{12.73 \times 1}{1.14 \times 10^{-5}} = 9 \times 10^5
\]
Ex. (10.68)

Given: Commercial steel pipe to carry 300 cfs of water at 60°F with a head loss of 1 ft per 1000 ft of pipe. Assume pipe sizes are available in even sizes when the diameters are expressed in inches (i.e., 10 in, 12 in, etc.).

Find: Diameter.

Solution: \( \nu = 1.22 \times 10^{-5} \text{ ft}^2 / \text{s} \); \( k_s = 1.5 \times 10^{-4} \text{ ft} \)

Assume \( f = 0.015 \)

\[
h_f = f \frac{L \nu^2}{D^2 g}
\]

\[
1 = 0.015 \times \frac{1000 (Q / (\pi / 4)D^2)^2}{D^5 2g}
\]

\[
1 = \frac{33.984}{D^5}
\]

\[
D = 8.06 \text{ ft}
\]

Relative roughness: \( \frac{k_s}{D} = \frac{1.5 \times 10^{-4}}{8.06} = 0.00002 \)

Get better estimate of \( f \)

\[
\text{Re} = \frac{VD}{\nu} = \frac{Q}{(\pi / 4)D^2} = \frac{Q}{(\pi / 4)D \nu}
\]

\[
\text{Re} = \frac{300}{(\pi / 4)(8.06)(1.22 \times 10^{-5})} = 3.9 \times 10^6
\]

\( f = 0.010 \)

\[
1 = \frac{22.656}{D^5}
\]

\[
D = 7.43 \text{ ft} = 89 \text{ in.}
\]

Use a 90 in pipe.
Ex. (10.81)

Given: The pressure at a water main is 300 kPa gage. What size pipe is needed to carry water from the main at a rate of 0.025 m³/s to a factory that is 140 m from the main? Assume galvanized-steel pipe is to be used and that the pressure required at the factory is 60 kPa gage at a point 10 m above the main connection.

Find: Size of pipe.

Solution:

\[ h_f = f \frac{L V^2}{D 2g} = f \frac{L (Q/(\pi/4)D^2)^2}{2g} \]

\[ D = \left( \frac{8 fL Q^2}{h_f \pi^2 g} \right)^{1/5} \]

Relative roughness: \[ \frac{k_s}{D} = \frac{0.15}{100} = 0.0015 \]

Friction factor: \[ f = 0.022 \]

\[ D = 0.100 \left( \frac{0.022}{0.020} \right)^{1/5} = 0.102 \text{ m} \]

Use 12 cm pipe

Assume \( f = 0.020 \)

\[ D = \left( \frac{8 fL Q^2}{h_f \pi^2 g} \right)^{1/5} = \left( \frac{8 \cdot 0.02 \cdot 140 \cdot (0.025)^2}{14.45 \cdot \pi^2 \cdot 9.81} \right)^{1/5} = 0.100 \text{ m} \]
Ex. (10.83)

Given: The 10-cm galvanized-steel pipe is 1000 m long and discharges water into the atmosphere. The pipeline has an open globe valve and 4 threaded elbows; \( h_1 = 3 \) m and \( h_2 = 15 \) m.

Find: What is the discharge, and what is the pressure at A, the midpoint of the line?

Solution:

\[
\alpha_1 \frac{V^2}{2g} + \frac{p_1}{\gamma} + z_1 - \sum h_L = \alpha_2 \frac{V^2}{2g} + \frac{p_2}{\gamma} + z_2
\]

\[
0 + 0 + 12 = (1 + K_e + K_v + 4K_b + f \frac{L}{D}) \frac{V^2}{2g} + 0 + 0
\]

\( D = 10 \)-cm and assume \( f = 0.025 \)

\[
24g = (1 + 0.5 + 10 + 4 \times 0.9 + 0.025 \frac{1000}{0.1}) V^2
\]

\[
V^2 = \frac{24g}{265.1}
\]

\( V = 0.942 \) m/s

\( Q = VA = 0.942(\pi/4)(0.10)^2 = 0.0074 \) m\(^3\)/s

\( Re = \frac{VD}{\nu} = \frac{0.942 \times 0.1}{1.31 \times 10^{-6}} = 7 \times 10^4 \)

So \( f = 0.025 \)

\[
\alpha_A \frac{V^2}{2g} + \frac{p_A}{\gamma} + z_A - \sum h_L = \alpha_2 \frac{V^2}{2g} + \frac{p_2}{\gamma} + z_2
\]

\[
\frac{p_A}{\gamma} + 15 = (2K_b + f \frac{L}{D}) \frac{V^2}{2g}
\]

\[
\frac{p_A}{\gamma} = (2 \times 0.9 + 0.025 \times 0.1) \frac{500}{0.1} \frac{(0.942)^2}{2g} - 15 = -9.6 \) m

\( p_A = 9810 \times (-9.26) = -90.8 \) kPa

Near cavitation pressure, not good!
Ex. (10.95)

Given: If the deluge through the system shown is 2 cfs, what horsepower is the pump supplying to the water? The 4 bends have a radius of 12 in and the 6-in pipe is smooth.

Find: Horsepower

Solution:

\[
\alpha_1 \frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 + h_p = \alpha_2 \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + \Sigma h_L
\]

\[
0 + 0 + 30 + h_p = 0 + 60 + \frac{V_2^2}{2g}(1 + 0.5 + 4K_b + f \frac{L}{D})
\]

\[
V = \frac{Q}{A} = \frac{2}{(\pi / 4)(1/2)^2} = 10.18 \text{ ft/s}
\]

\[
\frac{V_2^2}{2g} = 1.611 \text{ ft}
\]

\[
Re = \frac{VD}{\nu} = \frac{10.18*(1/2)}{1.22 \times 10^{-5}} = 4.17 \times 10^5
\]

So \(f = 0.0135\)

\[
h_p = 60 - 30 + 1.611(1 + 0.5 + 4 * 0.19 + 0.0135 \frac{1700}{(1/2)})
\]

\[
= 107.6 \text{ ft}
\]

\[
p = \frac{Qh_p}{550} = 24.4 \text{ hp}
\]