1. (25 Points) Consider the following initial value problem:

\[
\frac{dx}{dt} = -2t - x \quad x(t = 0) = -1, \quad 0 \leq t \leq 0.5, \quad \Delta t = 0.25
\]

(a) (10 Points) Use Euler’s method to solve the initial value problem.

(b) (10 Points) Use the Modified-Euler method to solve the initial value problem.

(c) (5 Points) Compute the percent relative error from parts (a) and (b) at \( t = 0.5 \) by comparing to the value of the exact solution

\[
x(t) = -3e^{-t} - 2t + 2
\]

(a) (10 Points) Use Euler’s method to solve the initial value problem.

Euler formula wrong (using exact solution formula, etc.), -3

\[
\frac{dx}{dt} = -2t - x \quad x(t = 0) = -1, \quad 0 \leq t \leq 0.5, \quad \Delta t = 0.25
\]

\[
x_{i+1} = x_i + \Delta t(-2t_i - x_i)
\]

\[
x_{0.25} = x_0 + 0.25(-2t_0 - x_0) = -1 + 0.25(-2*0 - (-1)) = -0.75
\]

\[
x_{0.5} = x_{0.25} + 0.25(-2t_{0.25} - x_{0.25}) = -0.75 + 0.25(-2*0.25 - (-0.75)) = -0.6875
\]

(b) (10 Points) Use the Modified-Euler method to solve the initial value problem.

Mod-Euler formula wrong, -3

Missing calculations, -5

\[
\frac{dx}{dt} = -2t - x \quad x(t = 0) = -1, \quad 0 \leq t \leq 0.5, \quad \Delta t = 0.25
\]

\[
x_{i+1}^* = x_i + \Delta t(-2t_i - x_i) \quad x_{i+1} = x_i + \frac{\Delta t}{2}(-2t_i - x_i - 2t_{i+1} - x_{i+1}^*)
\]

\[
x_{0.25}^* = x_0 + 0.25(-2t_0 - x_0) = -1 + 0.25(-2*0 - (-1)) = -0.75
\]
\[ x_{0.25} = x_0 + \frac{\Delta t}{2} (-2t_0 - x_0 - 2t_{0.25} - x_{0.25}) \]
\[ = -1 + \frac{0.25}{2} (-2*0 - (-1) - 2*0.25 - (-0.75)) \]
\[ = -0.8437 \]

\[ x^*_{0.5} = x_{0.25} + 0.25(-2t_{0.25} - x_{0.25}) \]
\[ = -0.8437 + 0.25(-2*0.25 - (-0.8437)) \]
\[ = -0.758 \]

\[ x_{0.5} = x_{0.25} + \frac{\Delta t}{2} (-2t_{0.25} - x_{0.25} - 2t_{0.5} - x^*_{0.5}) \]
\[ = -0.8437 + \frac{0.25}{2} (-2*0.25 - (-0.8437) - 2*0.5 - (-0.758)) \]
\[ = -0.831 \]

(c) (5 Points) Compute the percent relative error from parts (a) and (b) at \( t = 0.5 \) by comparing to the value of the exact solution

\[ x(t) = -3e^{-t} - 2t + 2 \]

\[ x(0.5) = -3e^{-0.5} - 2*0.5 + 2 \]
\[ = -3*0.6065 - 2*0.5 + 2 \]
\[ = -0.8195 \]

Euler \( x_{0.5} = -0.6875 \)

\[ Error = \left| \frac{x(0.5) - x_{0.5}}{x(0.5)} \right| *100 \]
\[ = \left| \frac{-0.8195 - (-0.6875)}{-0.8195} \right| *100 \]
\[ = 16.12\% \]

Mod-Euler \( x_{0.5} = -0.831 \)

\[ Error = \left| \frac{x(0.5) - x_{0.5}}{x(0.5)} \right| *100 \]
\[ = \left| \frac{-0.819 - 0.831}{-0.819} \right| *100 \]
\[ = 1.40\% \]
2. (10 points) Find the products $AB$ and $BA$ if they are defined:

Incorrect matrix manipulation, -5

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 3 & 1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 4 \\ 8 \\ 9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 3 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 1*1+2*4+4*8+5*9 \\ 3*1+1*4+0*8+9*2 \end{bmatrix} = \begin{bmatrix} 86 \\ 25 \end{bmatrix}$$

$BA$ is undefined

3. (15 points) Solve the following system of equations by Gauss elimination. Show your work.

Not knowing method and unable to start problem, -15
Not knowing method (using Gauss Seidel, etc.), -10
Only getting through part of steps, -10
Getting through a few more of the steps, -5

$$-12x_1 + x_2 - 8x_3 = -80$$
$$x_1 - 6x_2 + 4x_3 = 13 \quad (0.12)$$
$$-2x_1 - x_2 + 10x_3 = 90 \quad (-0.6)$$

$$-12x_1 + x_2 - 8x_3 = -80$$
$$0 - 71x_2 + 40x_3 = 76 \quad (0.7)$$
$$0 + 7x_2 - 68x_3 = -620 \quad (0.71)$$

$$-12x_1 + x_2 - 8x_3 = -80$$
$$0 - 71x_2 + 40x_3 = 76$$
$$0 + 0 - 4548x_3 = -43488$$

$$x_3 = 9.562$$
$$x_2 = 4.3166$$
$$x_1 = 0.65171$$
4. (15 points) Start from an initial guess of \( x_1^0 = x_2^0 = x_3^0 = 0.0 \), show the first 2 iterations of the Gauss-Seidel method for the solution of the system of equations in problem 3. Compute the error after the second iteration.

Not knowing method and unable to start problem, -15
Using Jacobi, -5
No error, -5

\[
\begin{align*}
-12x_1 + x_2 - 8x_3 &= -80 \\
x_1 - 6x_2 + 4x_3 &= 13 \\
-2x_1 - 2x_2 + 10x_3 &= 90
\end{align*}
\]

\( x_1^{i+1} = \frac{-80 - (x_2^i - 8x_3^i)}{-12} \)

\( x_2^{i+1} = \frac{13 - (x_1^{i+1} + 4x_3^i)}{-6} \)

\( x_3^{i+1} = \frac{90 - (-2x_1^{i+1} - x_2^{i+1})}{10} \)

\( x_1^1 = \frac{-80}{-12} = 6.6667 \)

\( x_2^1 = \frac{13 - (6.6667)}{-6} = -1.0556 \)

\( x_3^1 = \frac{90 - (-2 * 6.6667 + 1.0556)}{10} = 10.2278 \)

\( x_1^2 = \frac{-80 - (-1.0556 - 8 * 10.2278)}{-12} = -0.2398 \)

\( x_2^2 = \frac{13 - (-0.2398 + 4 * 10.2278)}{-6} = 4.61191 \)

\( x_3^2 = \frac{90 - (-2 * -0.2398 - 4.61191)}{10} = 9.41323 \)

\( e_1 = \left| \frac{0.2398 - 6.6667}{0.2398} \right| * 100 = 2880\% \)

\( e_2 = \left| \frac{4.61191 + 1.05556}{4.61191} \right| * 100 = 122\% \)

\( e_3 = \left| \frac{9.41323 - 10.2278}{9.41323} \right| * 100 = 8.6\% \)
5. (15 points) Given the following nonlinear equation:

\[ 0.9x^2 - 1.7x - 2.5 = 0 \]

(a) (7 points) Write the equation for first iteration of Fixed point iteration
Not knowing method and unable to start problem, -7
Not knowing method, -5

\[ x_{i+1} = g(x_i) \]

\[ x_{i+1} = \frac{-2.5 + 0.9x_i^2}{1.7} \]

(b) (8 points) Write the equation for first iteration of Newton’s method
Not knowing method and unable to start problem, -8
Not knowing method, -5

\[ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \]

\[ x_{i+1} = x_i - \frac{0.9x_i^2 - 1.7x_i - 2.5}{1.8x_i - 1.7} \]

6. (20 points) Given the following integral

\[ I = \int_0^{3\pi/2} \sin(4x + 2)dx \]

(a) (6 points) Evaluate the above integral analytically

\[
\begin{align*}
I &= \int_0^{3\pi/2} \sin(4x + 2)dx \\
&= -\frac{\cos(4x + 2)}{4} \bigg|_0^{3\pi/2} \\
&= -\frac{\cos(6\pi + 2) - \cos(2)}{4} \\
&= 0
\end{align*}
\]

(b) (7 points) Using points running from \( i = 0 \) to \( i = n \), write the equation for evaluating the integral numerically using Trapezoid rule, and

Picking a number of points when none was specified, -3
Getting formula wrong, -2
Summing 0 to n instead of 1 to n-1, -1
Not identifying \( f(x) \) in formula, -2

\[
I = \left[ \sin(4x + 2) \right]_{0}^{3\pi/2} \approx \frac{\Delta x}{2} \left[ \sin(4 \cdot 0 + 2) + 2 \sum_{i=1}^{n-1} \sin(4x_i + 2) + \sin(4 \cdot 3\pi / 2 + 2) \right]
\]

(7 points) Using points running from \( i = 0 \) to \( i = n \), write the equation for evaluating the integral numerically using Simpson’s 1/3 Rule.

Not getting odds and evens correct, -2

Getting formula wrong, -2

\[
I = \left[ \sin(4x + 2) \right]_{0}^{3\pi/2} \approx \frac{\Delta x}{3} \left[ \sin(4 \cdot 0 + 2) + 4 \sum_{i=1}^{n-1} \sin(4x_i + 2) + 2 \sum_{i=1}^{n-1} \sin(4x_i + 2) + \sin(4 \cdot 3\pi / 2 + 2) \right]
\]

\( i \) odd, \( i = 1 \) to \( n-1 \)

\( i \) even, \( i = 1 \) to \( n-1 \)