Numerical Methods for Civil Engineers

Lecture Notes
CE 311K
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Matrices

Introduction

An important tool in many areas of scientific and engineering analysis and computation is matrix theory or linear algebra. A wide variety of problems lead ultimately to the need to solve a linear system of equations $\boldsymbol{Ax} = \boldsymbol{b}$. There are two general approaches to the solution of linear systems. Direct methods, such as the Gauss elimination method, or iterative methods, such as the Gauss Siedel method. In this section, we will discuss some of the basic notation and results of linear algebra and then explore their use in the solution of linear systems of equations by Gauss elimination.

Matrix Notation

A matrix is defined to be a rectangular array of numbers arranged into rows and columns. A matrix with *m* rows and *n* columns can be written as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

This array is an (m x n) matrix. A matrix has no numerical value, it is just a way of representing arrays (or tables) of numbers. The horizontal elements of the matrix are the *rows* and the vertical elements are the *columns*. Since this matrix is a two-dimensional array of numbers a double subscript is used to locate any element in the matrix. The first subscript of an element designates the row, and the second subscript designates the column. Any matrix which has the same number of rows as columns is called a square matrix.

Examples. The following are matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & e \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

But the following are not, since they are not rectangular arrays arranged into rows and columns:

$$\begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix} \qquad \begin{bmatrix} & 4 & \\ 5 & & 6 \end{bmatrix} \qquad \begin{bmatrix} & 11 & \\ & 7 & \\ 8 & & 9 \\ & 10 & \end{bmatrix}$$

Matrices are usually denoted by uppercase bold letters, e.g., $\boldsymbol{A}$, $\boldsymbol{B}$, …, and elements of matrices by lowercase italic letters, e.g., a_{ij}, b_{ij}, …

A *row matrix* (or *row vector*) is a matrix with one row, i.e., the dimension $m = 1$. For example

$$\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & r_3 & \cdots & r_n \end{pmatrix}$$

A *column vector* is a matrix with only one column, e.g.,

$$\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

When the row and column dimensions of a matrix are equal ($m = n$) then the matrix is called <u>*square*</u>

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The <u>*transpose*</u> of the (m x n) matrix $\boldsymbol{A}$ is the (n x m) matrix formed by interchanging the rows and columns such that row i becomes column i of the transposed matrix

$$\boldsymbol{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & & \ddots & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Examples:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \qquad \boldsymbol{A}^T = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix} \qquad \boldsymbol{A}^T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

A <u>*symmetric*</u> matrix is one where $\boldsymbol{A}^T = \boldsymbol{A}$.

Example: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Matrix Arithmetic

Two (m x n) matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are *equal* if and only if each of their elements are equal. That is

$$\boldsymbol{A} = \boldsymbol{B} \text{ if and only if } a_{ij} = b_{ij} \qquad \text{for } i = 1,...,m \text{ and } j = 1,...,n$$

The *addition* of vectors and matrices is allowed whenever the dimensions are the same. The sum of two (m x 1) column vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_m + b_m \end{pmatrix}$$

Examples:

Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 3 \\ 5 \\ -1 \\ -2 \end{bmatrix}$. Then

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1+3 \\ -3+5 \\ 2-1 \\ 4-2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$5\boldsymbol{u} = 5\begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \\ 10 \\ 20 \end{bmatrix}$$

$$2\boldsymbol{u}-3\boldsymbol{v}=2\begin{bmatrix}1\\-3\\2\\4\end{bmatrix}-3\begin{bmatrix}3\\5\\-1\\-2\end{bmatrix}=\begin{bmatrix}2\\-6\\4\\8\end{bmatrix}+\begin{bmatrix}-9\\-15\\3\\6\end{bmatrix}=\begin{bmatrix}2-9\\-6-15\\4+3\\8+6\end{bmatrix}=\begin{bmatrix}-7\\-21\\7\\14\end{bmatrix}$$

The sum of two (m x n) matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & & a_{1n}\\ \vdots & & \ddots & \vdots\\ a_{m1} & a_{m2} & \dots & a_{mn}\end{pmatrix}+\begin{pmatrix}b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & & b_{1n}\\ \vdots & & \ddots & \vdots\\ b_{m1} & b_{m2} & \dots & b_{mn}\end{pmatrix}$$

$$=\begin{pmatrix}a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n}\\ a_{21}+b_{21} & a_{22}+b_{22} & & a_{1n}+b_{1n}\\ \vdots & & \ddots & \vdots\\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn}\end{pmatrix}$$

Examples:

1. $\boldsymbol{A}=\begin{bmatrix}1 & 2 & 3\\2 & 1 & 4\\1 & 4 & 3\end{bmatrix}$ $\quad\boldsymbol{B}=\begin{bmatrix}3 & 2 & 1\\-4 & 1 & 2\\2 & 3 & 1\end{bmatrix}$

$$\boldsymbol{A}+\boldsymbol{B}=\begin{bmatrix}1 & 2 & 3\\2 & 1 & 4\\1 & 4 & 3\end{bmatrix}+\begin{bmatrix}3 & 2 & 1\\-4 & 1 & 2\\2 & 3 & 1\end{bmatrix}=\begin{bmatrix}4 & 4 & 4\\-2 & 2 & 6\\3 & 7 & 2\end{bmatrix}$$

2. The following matrix addition is ***not defined***:

$$\begin{bmatrix}1 & 2\\5 & 2\end{bmatrix}+\begin{bmatrix}2 & 4 & 6\\1 & 3 & 5\end{bmatrix}=?\quad\text{(not defined!)}$$

3. $\boldsymbol{A}=\begin{bmatrix}2 & 5\\3 & 6\\5 & 8\end{bmatrix}$ $\quad\boldsymbol{B}=\begin{bmatrix}4 & 2\\1 & 0\\2 & 0\end{bmatrix}$

$$\boldsymbol{A}-\boldsymbol{B}=\begin{bmatrix}2 & 5\\3 & 6\\5 & 8\end{bmatrix}-\begin{bmatrix}4 & 2\\1 & 0\\2 & 0\end{bmatrix}=\begin{bmatrix}-2 & 3\\2 & 6\\2 & 8\end{bmatrix}$$

<u>*Multiplication*</u> of a matrix $\boldsymbol{A}$ by a scalar α is defined as

$$\alpha\boldsymbol{A} = \begin{pmatrix}\alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n}\\ \alpha a_{21} & \alpha a_{22} & & \alpha a_{2n}\\ \vdots & & \ddots & \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn}\end{pmatrix}$$

Examples:

$$\alpha = 4,\quad \boldsymbol{A}=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix},\quad \alpha\boldsymbol{A}=\begin{bmatrix}4 & 8\\0 & 4\end{bmatrix}$$

$$\alpha = -2,\qquad \boldsymbol{A}=\begin{bmatrix}1 & 4 & 1\\2 & 0 & 3\end{bmatrix},\qquad \alpha\boldsymbol{A}=\begin{bmatrix}-2 & -8 & -2\\-4 & 0 & -6\end{bmatrix}$$

The <u>*product*</u> of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined only if the number of columns of $\boldsymbol{A}$ is equal to the number of rows of $\boldsymbol{B}$. If $\boldsymbol{A}$ is (m x p) and $\boldsymbol{B}$ is (p x n), the product is an (m x n) matrix $\boldsymbol{C}$

$$\boldsymbol{C} = \boldsymbol{AB} = \begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1p}\\ a_{21} & a_{22} & & a_{1p}\\ \vdots & & \ddots & \vdots\\ a_{m1} & a_{m2} & \dots & a_{mp}\end{pmatrix}\begin{pmatrix}b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & & b_{1n}\\ \vdots & & \ddots & \vdots\\ b_{p1} & b_{p2} & \dots & b_{pn}\end{pmatrix}$$

$$= \begin{pmatrix}a_{11}b_{11}+\cdots+a_{1p}b_{p1} & a_{11}b_{12}+\cdots+a_{1p}b_{p2} & \cdots & a_{11}b_{1n}+\cdots+a_{1p}b_{pn}\\ a_{21}b_{11}+\cdots+a_{2p}b_{p1} & a_{21}b_{12}+\cdots+a_{2p}b_{p2} & & a_{21}b_{1n}+\cdots+a_{2p}b_{pn}\\ \vdots & & \ddots & \vdots\\ a_{m1}b_{11}+\cdots+a_{mp}b_{p1} & a_{m1}b_{12}+\cdots+a_{mp}b_{p2} & \dots & a_{m1}b_{1n}+\cdots+a_{mp}b_{pn}\end{pmatrix}$$

The *ij* element of the matrix $\boldsymbol{C}$ is given by

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

That is the c_{ij} element is obtained by adding the products of the individual elements of the *i*th *row* of the first matrix by the *j*th *column* of the second matrix (i.e., "row-by-column"). The following figure shows an easy way to check if two matrices are compatible for multiplication and what the dimensions of the resulting matrix will be:

$$\boldsymbol{C}_{mxn} = \boldsymbol{A}_{mxp} \boldsymbol{B}_{pxn}$$

Examples:

1. Let $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, then

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_1, a_2, \cdots, a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

2. Let $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, then

$$\boldsymbol{Ab} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{matrix} a_{11}b_1 + a_{12}b_2 + \cdots a_{1n}b_n \\ a_{11}b_1 + a_{22}b_2 + \cdots a_{2n}b_n \\ \\ a_{n1}b_1 + a_{n2}b_2 + \cdots a_{nn}b_n \end{matrix}$$

3. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\quad \boldsymbol{B} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$

$$\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 3 \cdot 3 & 1 \cdot 1 + 3 \cdot 5 \\ 2 \cdot 2 + 4 \cdot 3 & 2 \cdot 1 + 4 \cdot 5 \end{bmatrix} = \begin{bmatrix} 11 & 16 \\ 16 & 22 \end{bmatrix}$$

4. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 4 & 3 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 2 & 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 1 \\ 2 \cdot 2 + 1 \cdot 1 + 4 \cdot 2 & 2 \cdot 1 + 1 \cdot 2 + 4 \cdot 1 \\ 1 \cdot 2 + 4 \cdot 1 + 3 \cdot 2 & 1 \cdot 1 + 4 \cdot 2 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 13 & 8 \\ 12 & 12 \end{bmatrix}$$

5. Let $\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ 5 & 2 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 4 & 7 \\ 6 & 8 \end{bmatrix}$

$$\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 12 & 21 \\ 10 & 15 \\ 32 & 51 \end{bmatrix}$$

A *diagonal* matrix is a square matrix where elements off the main diagonal are all zero

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$

The number one in the real number system has the property that for any other number a, 1a=a1=a; also 1(1) = 1. A matrix with these properties is called the identity matrix. An *identity* matrix is a square matrix where all the elements on the main diagonal are ones and every other element is zero

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Matrix <u>*division*</u> is not a defined operation. The identity matrix has the property that if $\boldsymbol{A}$ is a square matrix, then

$$\boldsymbol{IA} = \boldsymbol{AI} = \boldsymbol{A}$$

If $\boldsymbol{A}$ is an (n x n) square matrix and there is a matrix $\boldsymbol{X}$ with the property that

$$\vec{\boldsymbol{A}}\boldsymbol{X} = \boldsymbol{I}$$

where $\boldsymbol{I}$ is the identity matrix, then the matrix $\boldsymbol{X}$ is defined to be the *inverse* of $\boldsymbol{A}$ and is denoted $\boldsymbol{A}^{-1}$. That is

$$\boldsymbol{AA}^{-1} = \boldsymbol{I} \qquad \text{and} \qquad \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

The inverse of a (2 x 2) matrix $\boldsymbol{A}$ can be represented simply as

$$\boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example:

If $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, then $\boldsymbol{A}^{-1} = \frac{1}{2(2) - 1(1)} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$

An <u>*upper triangular*</u> matrix is one where all the elements below the main diagonal are zero

$$
\boldsymbol{U} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ 0 & 0 & \ddots & 0 \\ 0 & 00 & 0 & a_{nn} \end{pmatrix}
$$

A *lower triangular* matrix is one where all the elements above the main diagonal are zero

$$
\boldsymbol{L} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \vdots & & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}
$$

Exercises

1. When addition is defined, add the matrices A and B in the following cases:

   a. $\boldsymbol{A} = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 1 & 6 \end{bmatrix}$ $\quad \boldsymbol{B} = \begin{bmatrix} 9 & 7 & 2 \\ 0 & 1 & 8 \end{bmatrix}$

   b. $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \end{bmatrix}$ $\quad \boldsymbol{B} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

   c. $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

   d. $\boldsymbol{A} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ $\quad \boldsymbol{B} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

   e. $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 6 \end{bmatrix}$ $\quad \boldsymbol{B} = \begin{bmatrix} 7 & 4 \\ 1 & 0 \end{bmatrix}$

   f. $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. Find the products AB and BA when they are defined:

a. $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

b. $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 3 & 1 & 0 & 2 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 1 \\ 4 \\ 8 \\ 9 \end{bmatrix}$

c. $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 7 \\ 5 & 6 & 9 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 10 & 2 & 0 \\ 7 & 1 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

d. $\boldsymbol{A} = \begin{bmatrix} 1 \\ 0 \\ 7 \\ 8 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 2 & 4 & 9 & 6 & 5 & 0 \end{bmatrix}$

e. $\boldsymbol{A} = \begin{bmatrix} 1 \\ 0 \\ 7 \\ 8 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 2 & 4 & 9 & 6 \end{bmatrix}$

f. $\boldsymbol{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3. Given the diagonal matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, compute $\boldsymbol{AB}$ and $\boldsymbol{BA}$. What is true of $\boldsymbol{AB}$ and $\boldsymbol{BA}$?

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & b_{nn} \end{bmatrix}$$

4. Given the matrices

$$X=\begin{bmatrix}1 & 6\\ 3 & 10\\ 7 & 4\end{bmatrix} \quad Y=\begin{bmatrix}6 & 0\\ 1 & 4\end{bmatrix} \quad Z=\begin{bmatrix}1 & 1\\ 6 & 8\end{bmatrix}$$

a. Perform all the multiplications: ***X*Y***, ***X*Z***, ***Y*Z***, ***Z*Y***, ***Y*X***, ***Z*X***

b. Show that ***Y*Z*** is not equal to ***Z*Y***

5. A number of matrices are defined as

$$A=\begin{bmatrix}4 & 5\\ 1 & 2\\ 5 & 6\end{bmatrix} \quad B=\begin{bmatrix}4 & 3 & 7\\ 1 & 2 & 6\\ 1 & 0 & 4\end{bmatrix} \quad C=\begin{Bmatrix}2\\ 6\\ 1\end{Bmatrix}$$

$$D=\begin{bmatrix}5 & 4 & 3 & 6\\ 2 & 1 & 7 & 5\end{bmatrix} \quad E=\begin{bmatrix}1 & 5 & 6\\ 7 & 1 & 3\\ 4 & 0 & 5\end{bmatrix} \quad F=\begin{bmatrix}2 & 0 & 1\\ 1 & 6 & 3\end{bmatrix} \quad G=\begin{bmatrix}8 & 6 & 4\end{bmatrix}$$

a. State the dimensions of each matrix.
b. State if each matrix is Square, Rectangular, Column, or Row
c. What are the values of A_{12}; B_{23}; D_{32}; E_{22}; F_{12}; G_{12}?
d. Perform the following operations, if defined:

(1) A + B	(2) B – A	(3) A + F	(4) 5 * *B*
(5) A*B	(6) B*A	(7) A*C	(8) C^T
(9) D^T	(10) I*B		

6. Is the identity matrix a matrix all of whose elements are equal to one? Yes __X__, No _____

7. If $A=\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}$, then $C=\begin{bmatrix}17 & 16\\ 7 & 6\end{bmatrix}$ denotes the

a) __X__ product of A and B; b) _____ sum of A and B; c) _____ difference of A and B

Arrays

8. The following examples each involve references to arrays. Describe the array that is referred to in each situation.

a.
```
Dim Cost(100) As Single, Items(100,3) As Integer
```

b.
```
P(i) = P(i) + Q(i,j)
```

c.
```
Dim Sum As Double
…
Sum = 0
For k = 0 To 3
     For j = 0 To 2
          Sum = Sum + Z(k,j)
     Next j
Next k
```

9. Shown below are several statements or groups of statements involving arrays or array elements. Describe the purpose of each statement or group of statements.

a.
```
Dim Values(12) As Single
…
Call Sub1(Values(3))
```

b.
```
Dim Values(12) As Single
…
Call Sub1(Values())
```
(Compare with the previous question)

10. Write one or more Visual Basic statements for each of the following situations. Assume that each subscript ranges from 1 to its maximum value (rather than from 0 to its maximum value).

a. Sum the first `n` elements of the one-dimensional array `Costs( )`

b. Sum all elements in column 3 of the two-dimensional array `Values`(60,20).

c. Sum all elements in the first `m` rows of the two-dimensional array `Values(60,20)` described in Part b.

d. Calculate the square root of the sum of the squares of the first 100 odd elements of the one-dimensional array `X`(200); i.e., calculate

$$\sqrt{X(1)^2 + X(3)^2 + X(5)^2 + \cdots + X(199)^2}$$

e. A two-dimensional array `W(k,k)` has `k` rows and `k` columns. Calculate the product of the elements on the main diagonal of `W`(,).