Numerical Methods for Civil Engineers

Lecture Notes
CE 311K
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Numerical Integration

Introduction

Trapezoid Rule

The primary purpose of numerical integration (or quadrature) is the evaluation of integrals which are either impossible or else very difficult to evaluate analytically. Numerical integration is also essential for the evaluation of integrals of functions available only at discrete points. Such functions often arise in the numerical solution of differential equations or from experimental data taken at discrete intervals.

Consider an integrable function *f(x)* on the interval $a \le b$. We wish to evaluate the integral

$$I = \int_a^b f(x)dx$$

We divide the interval $a \le b$ into *n* equal subintervals each of width Δx, where

$$\Delta x = \frac{b - a}{n}$$

The function and the division into subintervals are shown in Figure 1. Each of the subintervals will be referred to as a *panel*. Consider an expanded view of a general region including two panels as shown in Figure 2. In Figure 2, the points $f(x_{i-1})$, $f(x_i)$, and $f(x_{i+1})$ have been connected by straight lines. These straight lines approximate the function $f(x)$ between x_{i-1} and x_i, x_i and x_{i+1}, etc.

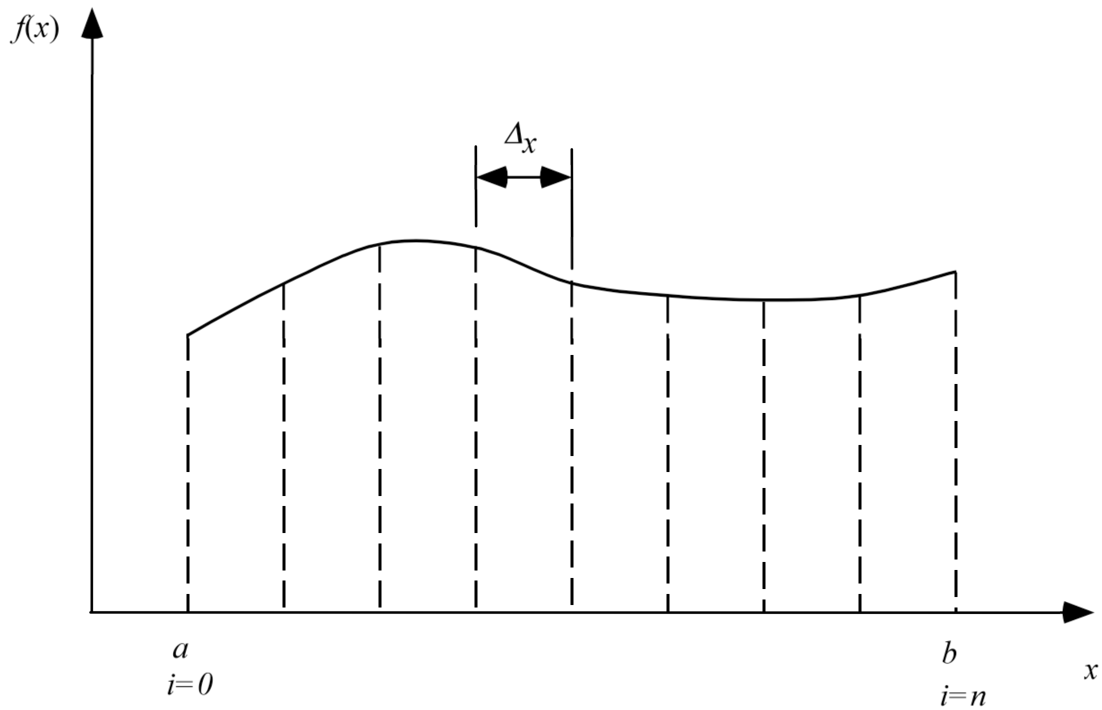

Figure 1. Division of interval of integration into a number of panels.

Approximating the area of each panel (with endpoints x_{i-1} and x_i) by the area under the straight lines yields

$$\int_{x_{i-1}}^{x_i} f(x)dx \approx \frac{f_{i-1} + f_i}{2}(\Delta x)$$

and

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \frac{f_i + f_{i+1}}{2}(\Delta x)$$

Since the true area of each panel has been approximated by the area of a trapezoid, this approximate numerical evaluation of an integral is called the *trapezoidal rule*. Geometrically, the trapezoidal rule is equivalent to approximating the area of the trapezoid under the straight line connecting the two endpoints of each panel.

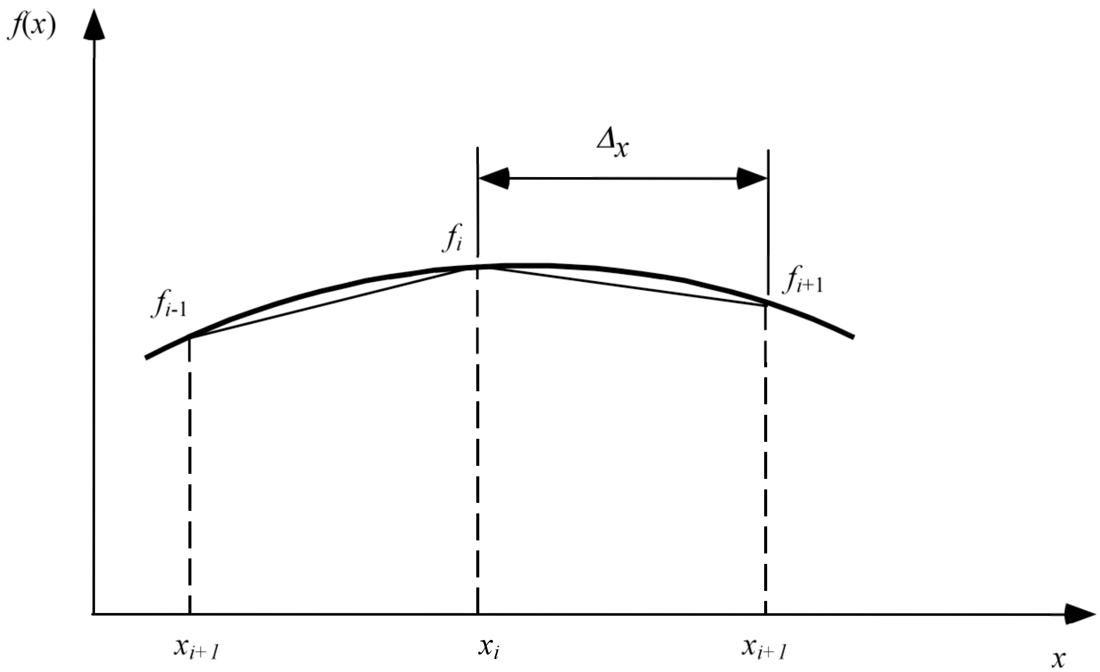

Figure 2. View of 2 panels.

We have employed the subscript notation that $f(x_i) = f_i$. The integral over the two panels is given by

$$\int_{x_{i-1}}^{x_{i+1}} f(x)dx = \int_{x_{i-1}}^{x_i} f(x)dx + \int_{x_i}^{x_{i+1}} f(x)dx$$

From equations (3) and (4), this integral can be approximated by

$$\int_{x_{i-1}}^{x_{i+1}} f(x)dx \approx \frac{f_i + f_{i+1}}{2}(\Delta x) + \frac{f_{i-1} + f_i}{2}(\Delta x) = \frac{\Delta x}{2}(f_{i-1} + 2f_i + f_{i+1})$$

By extending equation (6), the trapezoid rule approximation to the integral over the entire interval is

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n\text{-}2} + 2f_{n\text{-}1} + f_n)$$
$$= \frac{\Delta x}{2}(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n)$$

where $f_0 = f(a)$ and $f_n = f(b)$.

Reducing Δx will, in general, provide a more accurate representation of the integral, since a large number of short, connected, straight line segments can better approximate most functions than can a small number of long segments. An estimate of the error from the application of the trapezoidal rule over the interval between a and b is

$$E = -\frac{(\Delta x)^2}{12}(b\text{-}a)f''(\bar{x})$$

where $f''(\bar{x})$ is the average second derivative over the interval. Using equations (9) and (8), we have

$$I = \frac{\Delta x}{2}(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n) - \frac{(\Delta x)^2}{12}(b\text{-}a)f''(\bar{x})$$

The trapezoidal rule is termed a *second order* method of numerical integration because the error is proportional to $(\Delta x)^2$.

Example: Demonstrate the use of the trapezoidal rule with three panels to evaluate

$$I = \int_0^{\pi} \sin(x)dx$$

The trapezoidal rule using 3 panels yields

$$I = \frac{\pi/3}{2}\left[f(0)+2f(\pi/3)+2f(2\pi/3)+f(\pi)\right]$$
$$= \frac{\pi}{6}\left[\sin(0)+2\sin(\pi/3)+2\sin(2\pi/3)+\sin(\pi)\right]$$
$$= 0.523599\left[0+2(0.866025)+2(0.866025)+0\right]$$
$$= 1.813799$$

The correct answer is 2.0.

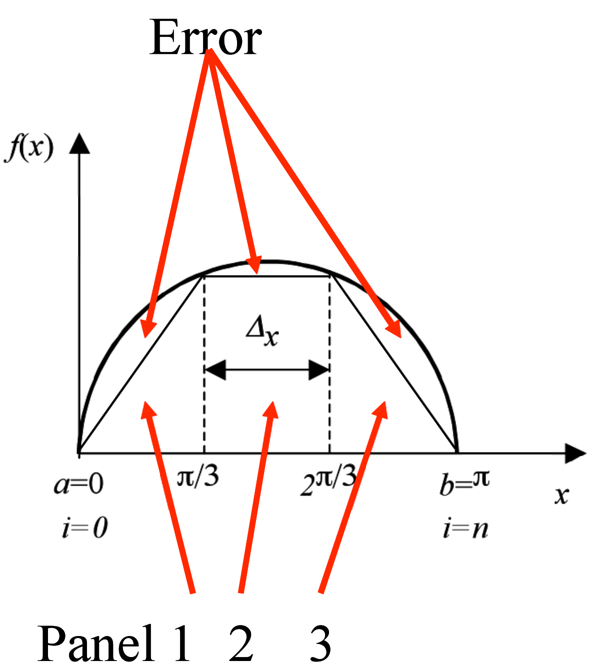

Figure 3. Example of Trapezoid Rule with 4 panels.

Simpson's Rule (1/3)

Simpson's rule is a numerical integration technique which is based on the use of parabolic arcs to approximate $f(x)$ instead of the straight lines employed in the trapezoid rule. Even higher order polynomials, such as cubics, can also be used to obtain more accurate results. Consider the integral of a function $f(x)$ over an interval $a \le b$

$$I = \int_a^b f(x)d$$

Simpson's 1/3 rule is obtained when a second-order interpolating polynomial is substituted for $f(x)$

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

where $f_2(x)$ is a second-order Lagrange interpolating polynomial using the three points x_{i-1}, x_i, and x_{i+1}

$$f_2(x) = \frac{(x - x_i)(x - x_{i+1})}{(x_{i-1} - x_i)(x_{i-1} - x_{i+1})} f(x_{i-1}) + \frac{(x - x_{i-1})(x - x_{i+1})}{(x_i - x_{i-1})(x_i - x_{i+1})} f(x_i)$$
$$+ \frac{(x - x_{i-1})(x - x_i)}{(x_{i+1} - x_{i-1})(x_{i+1} - x_i)} f(x_{i+1})$$

Consider an expanded view of a general region including one panel as shown in Figure 4 where the points $f(x_{i-1})$, $f(x_i)$, and $f(x_{i+1})$ have been connected by a parabola. This parabola approximates the function $f(x)$ between x_{i-1} and x_{i+1}.

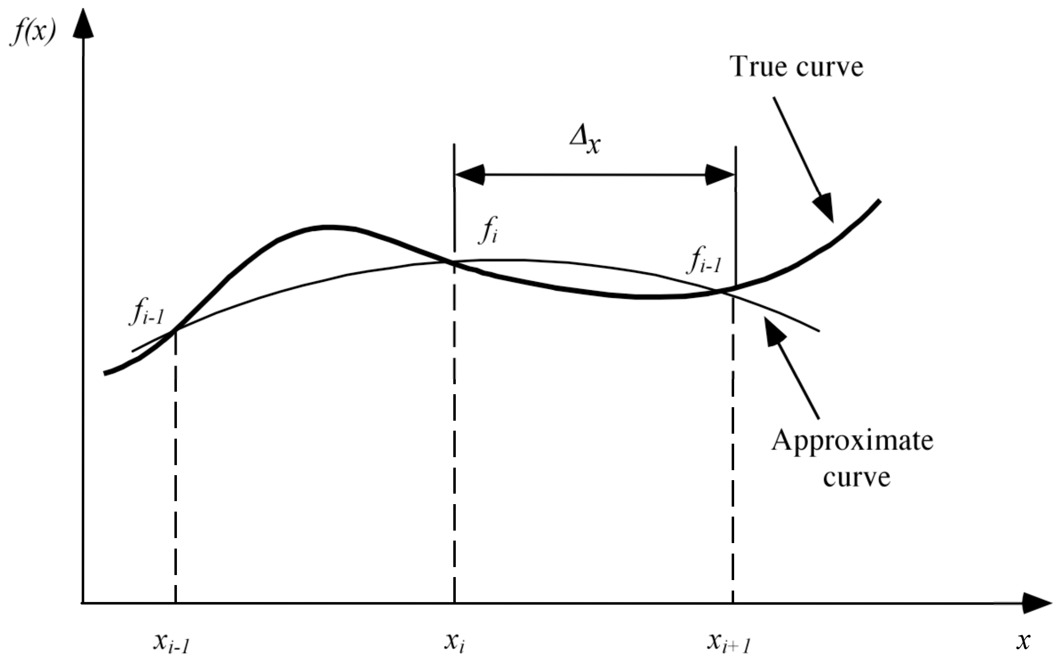

Figure 4. Two panels for Simpson's rule.

Approximating the area of the panel by the area under the parabola yields

$$\int_{x_{i-1}}^{x_{i+1}} f_2(x)dx = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} f(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})} f(x_i) + \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} f(x_{i+1}) \right] dx$$

This expression can be integrated and simplified to

$$\int_{x_{i-1}}^{x_{i+1}} f_2(x)dx = \frac{\Delta x}{3}\left[f(x_{i-1}) + 4f(x_i) + f(x_{i+1}) \right]$$

The Simpson's rule approximation to the integral over the entire interval is

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}(f_0+4f_1+2f_2+4f_3+\cdots+2f_{n-2}+4f_{n-1}+f_n)$$

$$= \frac{\Delta x}{3}(f_0+4\sum_{\substack{i=1\\ i\ odd}}^{n-1} f_i+2\sum_{\substack{i=2\\ i\ even}}^{n-2} f_i+f_n)$$

where $f_0 = f(x_0) = f(a)$ and $f_\mathrm{n} = f(x_\mathrm{n}) = f(b)$. The truncation error from the application of the Simpson's rule over the interval between a and b is

$$E = -\frac{(\Delta x)^4}{180}(b\text{-}a)f^{iv}(\bar{x})$$

where $f''(\bar{x})$ is the average second derivative over the interval. So, we have

$$\int_a^b f(x)dx = \frac{\Delta x}{3}(f_0+4\sum_{\substack{i=1\\ i\ odd}}^{n-1} f_i+2\sum_{\substack{i=2\\ i\ even}}^{n-2} f_i+f_n) - \frac{(\Delta x)^4}{180}(b\text{-}a)f^{iv}(\bar{x})$$

Simpson's rule is termed a *fourth order* method of numerical integration because the error is proportional to $(\Delta x)^4$.

Example: Demonstrate the use of the Simpson's rule with $n = 4$ to evaluate

$$I = \int_0^{\pi} \sin(x)dx$$

Simpson's rule with $n = 4$ yields

$$
\begin{aligned}
I &= \frac{\pi/4}{3}\left\{f(0)+4\left[f(\pi/4)+f(3\pi/4)\right]+2f(\pi/2)+f(\pi)\right\} \\
&= \frac{\pi}{12}\left\{\sin(0)+4\left[\sin(\pi/4)+\sin(3\pi/4)\right]+2\sin(\pi/2)+\sin(\pi)\right\} \\
&= 0.261799\left[0+4(0.707107+0.707107)+2(1.0)+0\right] \\
&= 2.00456
\end{aligned}
$$

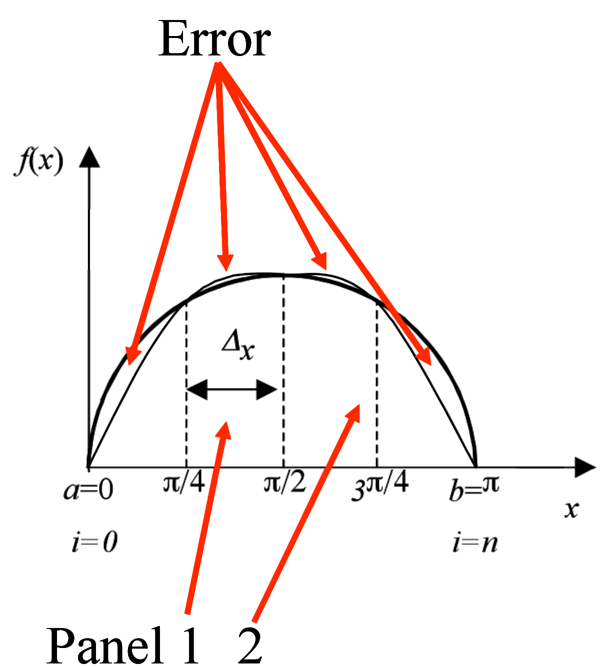

Figure 5. Example of Simpsons-1/3 Rule with 2 panels.

Multiple Integration

A double integral may be evaluated as an *iterated integral*; in other words, we may write

$$
I = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx = \int_a^b dx \int_{c(x)}^{d(x)} f(x,y)\,dy = \int_a^b g(x)\,dx
$$

where

$$
g(x) = \int_{c(x)}^{d(x)} f(x,y)\,dy
$$

Now we can evaluate *I* using any convenient numerical integration formula, say Simpson's rule rule.

$$I = \int_a^b g(x)dx \approx \frac{\Delta x}{3}\left[g(a) + 4\sum_{odd\, i} g(a+i\Delta x) + 2\sum_{even\, i} g(a+i\Delta x) + g(b)\right]$$

Example: $I = \int_2^3\int_2^3 (x^2 + y)dy\,dx$

$$I = \int_2^3 g(x)dx \quad g(x) = \int_2^3 f(x,y)dy \qquad f(x,y) = x^2 + y$$

$$I = \int_2^3 g(x)dx \approx \frac{\Delta x}{2}\left[g(2) + 2g(2.25) + 2g(2.5) + 2g(2.75) + g(3)\right]$$

$$\begin{aligned} g(x) &= \int_2^3 f(x,y)dy \\ &\approx \frac{0.25}{2}\left[f(x,2) + 2f(x,\ 2.25) + 2f(x,2.5) + 2f(x,2.75) + f(x,3)\right] \end{aligned}$$

$$\begin{aligned} g(2) &= \int_2^3 f(2,y)dy \\ &\approx \frac{0.25}{2}\left[f(2,2) + 2f(2,2.25) + 2f(2,2.5) + 2f(2,2.75) + f(2,3)\right] \\ &= 6.5 \end{aligned}$$

similarly

$$g(2.25) = \int_2^3 f(2.25,y)dy \approx 7.56$$

$$g(2.5) = \int_2^3 f(2.5,y)dy \approx 8.75$$

$$g(2.75) = \int_2^3 f(2.75, y)dy \approx 10.06$$

$$g(3) = \int_2^3 f(3, y)dy \approx 11.5$$

so

$$I = \int_2^3 g(x)dx \approx \frac{0.25}{2}\left[6.5 + 2(7.56) + 2(8.75) + 2(10.06) + 11.5\right] = 8.437$$

Exercises

1. Integrate the following function

$$I = \int_{-3}^{5} (4x+5)^3\, dx$$

(a) analytically
(b) Simpson's Rule using $n = 4$

2. Integrate the following function (a) analytically

$$I = \int_0^3 x e^{2x} dx$$

and numerically using $n = 4$
(b) Trapezoid Rule
(c) Simpson's Rule.
Compute the percent error for both numerical integrations using the analytical value as the true value.

3. Evaluate the integral of the following data using the Trapezoid Rule:

$$I = \int_0^{0.5} F(X)dX$$

X	0	0.1	0.2	0.3	0.4	0.5
$F(X)$	1	7	4	3	5	2

4. Integrate the following function using the Trapezoid Rule with n = 5:

$$\int_0^1 x^{0.1}(1.2 - x)(1 - e^{20(x-1)})dx$$

5. (a) What is the advantage or disadvantage of using Simpson's rule rather than the trapezoid rule to numerically integrate a function? (b)What is the major difference between Simpson's rule and the trapezoid rule?

6. How would you begin to numerically approximate the double integral:

$$I = \int_a^b \int_c^d f(x,y)dxdy$$

7. Given a set of N concentration measurements C_i at times t, set up the formulas for numerically integrating the following expression using the trapezoid rule:

$$\bar{t}_i = \frac{\int_0^T C_i t dt}{\int_0^T C_i dt}$$

8. Integration provides a means to compute how much mass enters or leaves a reactor over a specified time period, as in

$$M = \int_{t_1}^{t_2} Qcdt$$

where t_1 and t_2 are the initial and final times. This formula makes intuitive sense if you recall the analogy between integration and summation. Thus the integral represents the summation of the product of flow times concentration to give the total mass entering or leaving from t_1 to t_2. If the flow rate is constant, Q can be moved outside the integral. The outflow chemical concentration from a completely mixed reactor is measures as:

t, min	0	5	10	15	20	30	40	50	60

c, mg/m^3	10	20	30	40	60	80	70	50	60

For an outflow of $Q = 10$ m^3/min, use Simpson's 1/3 rule to estimate the mass of chemical that exits the reactor from $t = 0$ to 60 min. Solve the integral using Simpson's rule for the mass entering the reactor.

9. Integrate the function tabulated in the following table using the Trapezoid Rule.

X	f(x)	x	f(x)
1.6	4.953	2.8	16.445
1.8	6.050	3.0	20 .086
2.0	7.389	3.2	24.533
2.2	9.025	3.4	29.964
2.4	11.023	3.6	36.598
2.6	13.464	3.8	44.701

If the values in the table are from the exponential function $f(x) = e^x$, find the true value of the integral and compute the percent error in your integration.

10. Evaluate the following double integral (a) analytically, and (b) using the trapezoid rule with 4 panels.

$$I = \int_{-2}^{2}\int_{0}^{4}(x^2 - 3y^2 + xy^3)dxdy$$

Compute the percent error in your numerical calculation.

11. (a) Evaluate the following integral analytically.

(b) Evaluate the integral numerically using Trapezoid rule, with $n = 3$ panels (that is, 4 points).
(c) Compute the error between your analytical and approximate results.

$$I = \int_{0}^{3\pi/2} \sin(4x + 2)dx$$

12. Consider the following integral:

$$I = \int_{0}^{3\pi/2} \sin(5x + 1)dx$$

(a) Evaluate the integral analytically.

(b) Evaluate the integral numerically using Simpson's rule, with $n = 4$ panels (that is, 5 points).
(c) Compute the error between your analytical and approximate results.

13. Suppose you are an Architectural Engineer and you are planning to use a large parabolic arch with a shape given by:

$$y = 0.1x(30 - x)$$

where y is the height above the ground and x is in meters. Calculate the total length of the arch by using Simpson's 1/3 rule. (Divide the domain from x=0 to x=30 m into 10 equally spaced intervals.) The total length of the arc is given by

$$L = \int_{0}^{30} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

14. Evaluate the integral $I = \int_{0}^{2\pi} \cos^2 x dx$ using Simpson's 1/3 rule with 6 function evaluations.

Compute the error in your result. The exact value of the integral is $I = \int_{0}^{2\pi} \cos^2 x dx = \pi$