

Numerical Methods for Civil Engineers

Lecture Notes
CE 311K
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Errors and Stopping Criteria

Numerical methods involve approximations. There is always a discrepancy (or error) between the numerical and analytical solutions to any problem. Two major sources of error in numerical methods are:

Roundoff error: Computers represent quantities with a finite number of digits

Truncation error: Numerical methods employ approximations to represent exact mathematical operations and quantities

Computers represent and store numbers using a finite number of significant figures and omit the rest. The omission of significant figures, through either truncation or rounding, causes *roundoff error*. The result of roundoff error can become significant in cases where quantities of nearly equal magnitude are subtracted.

Error Estimates

The relationship between a true value (x) and an approximate value (\tilde{x}) is

$$\text{True value} = \text{Approximate value} + \text{Error}$$

or

$$x = \tilde{x} + E$$

where E is the error (discrepancy) between the true value and the approximate value

$$E = \text{Error} = \text{True value} - \text{Approximate value}$$

or

$$E = x - \tilde{x}$$

To account for the order of magnitude of the values being considered, we often normalize the error. The resulting relative error, e , is

$$e = \frac{|E|}{|x|} = \left| \frac{x - \tilde{x}}{x} \right|$$

We can relate relative error to the number of significant figures in the approximation by

$$e = \left| \frac{x - \tilde{x}}{x} \right| < \frac{1}{2} 10^{-d}$$

or

$$\begin{aligned} 2e &< 10^{-d} \\ \ln(2e) &< -d \ln(10) \\ d &< -\frac{\ln(2e)}{\ln(10)} \end{aligned}$$

Example. Pi is often approximated as: $\tilde{x} = 3.1416$ when a better approximation is $x = 3.1415927$. Find the error, relative error and the number of significant digits in the approximation.

$$\begin{aligned} E &= x - \tilde{x} \\ &= 3.1415927 - 3.1416 \\ &= -0.0000073 \end{aligned}$$

$$e = \left| \frac{-0.0000073}{3.1415927} \right|$$

$$= 0.0000023237$$

$$d < -\frac{\ln(2e)}{\ln(10)}$$

$$= -\frac{\ln(2 * 0.0000023237)}{\ln(10)}$$

$$= 5.3329$$

Thus, the approximation is good to 5 significant figures.

Often, we do not know the true value in our calculations, so we must use an alternative, approximate method to determine the error. In this method, we use our best available estimate, or approximation, of the true value. E_A is the approximate error between the current approximate value and our previous approximate value

$$E_A = \text{Approximate error} = \text{Present approximate value} - \text{Previous approximate value}$$

or

$$E_A = \tilde{x}^{k+1} - \tilde{x}^k$$

To account for the order of magnitude of the values being considered, we often normalize the error. The resulting relative approximate error, e_A , is

$$e_A = \left| \frac{E_A}{\tilde{x}^{k+1}} \right| = \left| \frac{\tilde{x}^{k+1} - \tilde{x}^k}{\tilde{x}^{k+1}} \right|$$

Example. The exponential function can be estimated by the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

As more terms are added to the sequence, the approximation becomes a better and better estimate of the value of $\exp(x)$. Estimate $\exp(x)$ for $x = 0.5$ by adding more and more terms to the sequence and computing the errors after adding each new term. Add terms until the estimate is valid to three significant digits.

$$e = \left| \frac{x - \tilde{x}}{x} \right| < \frac{1}{2} 10^{-2} = 0.005$$

Thus, we will add terms until the relative error falls below this value. We can get a true value from a calculator as $x = 1.648721271$

1. Using only the first term of the series, $e^x \approx 1$
2. Using the first and second terms of the series, $e^x \approx 1 + x = 1 + 0.5 = 1.5$
3. and so on ...

Considering the error associated with our approximations, we have

$$e = \left| \frac{x - \tilde{x}}{x} \right| = \left| \frac{1.648721271 - 1.5}{1.648721271} \right| = 0.09 = 9\%$$

# of Terms	Result	e
1	1	0.393
2	1.5	0.09
3	1.625	0.014
4	1.645833333	0.0017
5	1.648437500	0.00017
6	1.648697917	0.000014

Roundoff Errors

Computers retain only a fixed number of significant figures during a calculation. Numbers such as Π , e , or $\sqrt{3}$ can't be expressed by a finite number of digits. The discrepancy is called *roundoff error*. Within in the range of representable numbers on any computer, there is a finite number of quantities that can be represented. Approximation of number like (1) irrational numbers or (2) rational numbers that do not precisely match one of the values in the set of representable numbers must be approximated. There are two ways of accomplishing this approximation: (1) chopping and (2) rounding.

Example. Suppose $\pi = 3.14159265358\dots$ is stored on a computer with seven significant figures. One method of approximation is to simply *chop* off the eighth and higher terms i.e., $\pi = 3.141592$ with an error equal to

$$\begin{aligned} E &= \text{True value} - \text{Approximate value} \\ &= 3.14159265358\dots - 3.141592 \\ &= 0.00000065358\dots \end{aligned}$$

With chopping, any quantity falling within an interval Δx will be stored as the quantity at the lower end of the interval. Thus the upper bound on error for chopping is Δx . Note that the last digit discarded is 6. If rounding had been used, instead of chopping, our approximation would have been $\pi = 3.141593$, and the error would have been

$$E = 0.00000035\dots$$

Rounding yields a smaller error than chopping. The upper bound for error with rounding is $\Delta x/2$.

The number of significant figures carried on most computers allows most engineering calculations to be performed with more than acceptable precision. For example, the IBM PC

compatible family of computer (pre-1992) allowed 24 bits to be used for the mantissa which translates into about seven base-10 digits of precision.