CE 311 K - Introduction to Computer Methods

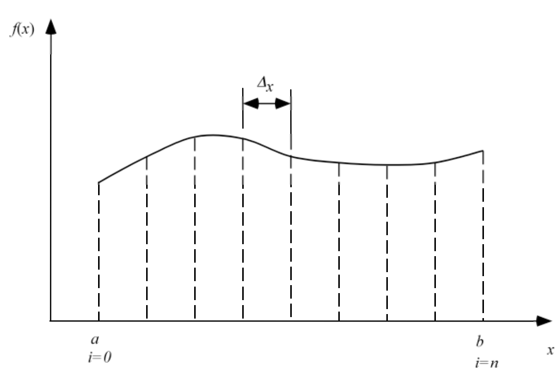
Numerical Integration

Numerical Integration

- Purpose is to evaluate integrals
  - Impossible or very difficult to evaluate analytically
  - Functions available only at discrete points
  - Divide the interval into n equal subintervals (panels) Δx wide

$$I = \int_a^b f(x)d$$

$$\Delta x = \frac{b-a}{n}$$

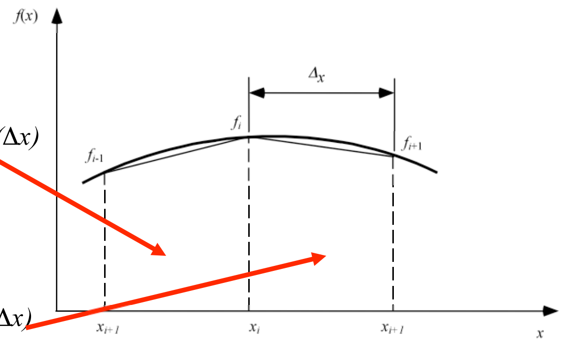


Trapezoid Rule

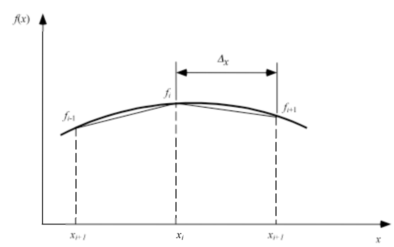
- Straight lines approximate the function between $f(x_{i-1})$ and $f(x_i)$, and $f(x_i)$ and $f(x_{i+1})$

$$\int_{x_{i-1}}^{x_i} f(x)dx \approx \frac{f_{i-1}+f_i}{2}(\Delta x)$$

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \frac{f_i+f_{i+1}}{2}(\Delta x)$$



Trapezoid Rule



- For the two panels, we have

$$\int_{x_{i-1}}^{x_{i+1}} f(x)dx = \int_{x_{i-1}}^{x_i} f(x)dx + \int_{x_i}^{x_{i+1}} f(x)dx$$
$$\approx \frac{\Delta x}{2}(f_{i-1}+f_i) + \frac{\Delta x}{2}(f_i+f_{i+1})$$
$$= \frac{\Delta x}{2}(f_{i-1}+2f_i+f_{i+1})$$

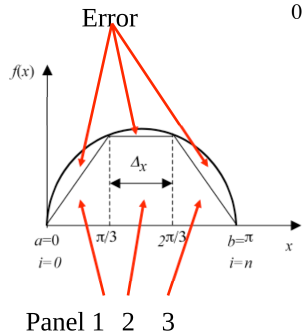
- Extending over entire interval

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(f_0+2f_1+\cdots+2f_{n-1}+f_n) = \frac{\Delta x}{2}\left(f_0+2\sum_{i=1}^{n-1} f_i + f_n\right)$$

Example - Trapezoid Rule

- Use trapezoidal rule with three panels to evaluate

$$I = \int_0^{\pi} \sin(x)dx$$

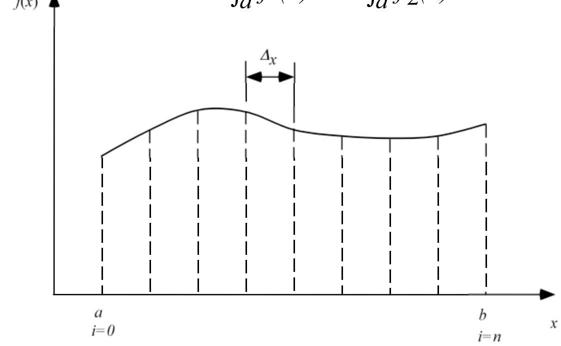


$$\begin{aligned} I &= \frac{\pi/3}{2}\left[f(0)+2f(\pi/3)+2f(2\pi/3)+f(\pi)\right] \\ &= \frac{\pi}{6}\left[\sin(0)+2\sin(\pi/3)+2\sin(2\pi/3)+\sin(\pi)\right] \\ &= 0.523599\left[0+2(0.866025)+2(0.866025)+0\right] \\ &= 1.813799 \end{aligned}$$

Simpson's 1/3 Rule

- Parabolic arcs are used to approximate the function between $f(x_{i-1})$ and $f(x_i)$, and $f(x_i)$ and $f(x_{i+1})$

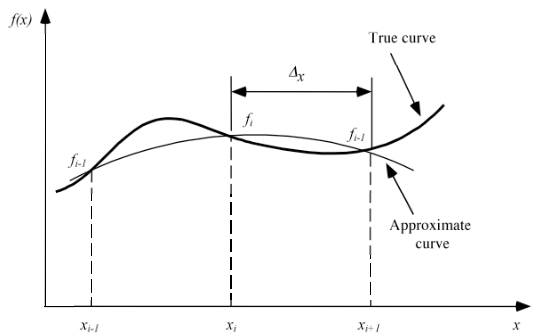
$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$



Simpson's 1/3 Rule

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

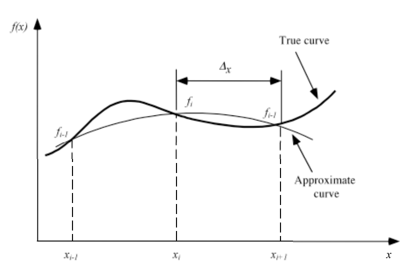
$$f_2(x) = \frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} f(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})} f(x_i) + \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} f(x_{i+1})$$



Simpson's 1/3 Rule

$$\int_{x_{i-1}}^{x_{i+1}} f_2(x)dx = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} f(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})} f(x_i) + \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} f(x_{i+1}) \right] dx$$

$$= \frac{\Delta x}{3} \left[f(x_{i-1}) + 4 f(x_i) + f(x_{i+1}) \right]$$



Simpson's 1/3 Rule

Extending this over the entire interval

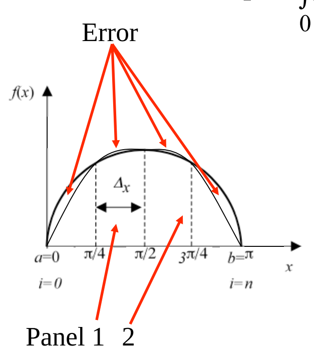
$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}(f_0+4f_1+2f_2+\cdots+4f_{n-1}+f_n)$$

$$= \frac{\Delta x}{3}(f_0+4\sum_{\substack{i=1 \\ i\,odd}}^{n-1} f_i+2\sum_{\substack{i=2 \\ i\,even}}^{n-2} f_i+f_n)$$

Example – Simpson's 1/3 Rule

- Use Simpson's rule with two panels to evaluate

$$I = \int_0^{\pi} \sin(x)dx$$



$$I = \frac{\pi/4}{3}\{f(0)+4[f(\pi/4)+f(3\pi/4)]+2f(\pi/2)+f(\pi)\}$$

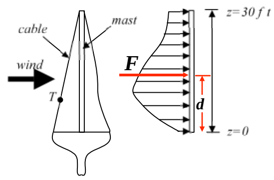
$$= \frac{\pi}{12}\{\sin(0)+4[\sin(\pi/4)+\sin(3\pi/4)]$$
$$+2\sin(\pi/2)+\sin(\pi)\}$$

$$= 0.2618[0+4(0.7071+0.7071)$$
$$+2(1.0)+0]$$

$$= 2.004$$

Example – Sailboat Mast

- Sailboat mast subject to distributed wind load. Total force resulting from wind load, *F*, given by:



$$F = \int_0^{30} 200\left(\frac{z}{5+z}\right)e^{-2z/30}\,dz$$

with line of action

$$d = \frac{\int_0^{30} 200z\left(\frac{z}{5+z}\right)e^{-2z/30}\,dz}{\int_0^{30} 200\left(\frac{z}{5+z}\right)e^{-2z/30}\,dz}$$

Find the tensile force *T* in the left support cable

Example – Trapezoid Rule

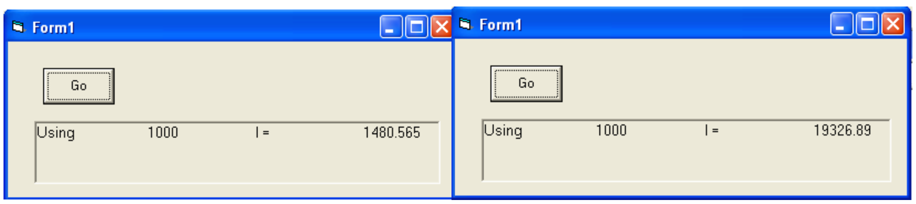
```
Private Sub cmdGo_Click()

    Dim panels As Integer
    Dim a As Single, b As Single, Integral As Single

    panels = 1000
    a = 0#
    b = 30#

    Integral = I(a, b, panels)
    picOutput.Print "Using ", panels, " the integral =  ", Integral

End Sub
```

```
Private Function I(a As Single, b As Single, n As Integer) As Single

    Dim sum As Single, x As Single, dx As Single
    Dim k As Integer
    sum = 0#
    dx = (b - a) / n
    For k = 2 To n
        x = a + dx * (k - 1)
        sum = sum + f(x)
    Next
    I = 0.5 * dx * (f(a) + 2 * sum + f(b))

End Function
```

Example – Trapezoid Rule

- Total Force, *F*, Line of Action, *d*

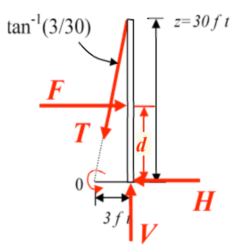
```
Private Function f(x As Single) As Single

'    f = 200# * (x / (5# + x)) * Exp(-2# * x / 30#)
     f = 200# * x * (x / (5# + x)) * Exp(-2# * x / 30#)

End Function
```



d = 19326.89/ 1480.57 = 13.05 ft

Example – Sailboat Mast

- Free body diagram



$$\sum F_H = 0 = F - T\sin\theta - H$$
$$\sum F_V = 0 = V - T\cos\theta$$
$$\sum M_0 = 0 = 3V - Fd$$

$$V = \frac{Fd}{3} = \frac{1480.6 * 13.05}{3} = 6440.6\,lb$$
$$T = \frac{V}{\cos\theta} = \frac{6440.6}{0.995} = 6473\,lb$$
$$H = F - T\sin\theta = 1480.6 - 6473 * 0.0995 = 836.54\,lb$$

Multiple Integrals

$$I = \int_a^b \left(\int_{c(x)}^{d(x)} f(x,y)dy \right) dx = \int_a^b g(x)dx$$

$$g(x) = \int_{c(x)}^{d(x)} f(x,y)dy$$

Sometimes *c*(*x*) and *d*(*x*) are constants, *c* and *d*

$$I = \int_a^b g(x)dx \approx \frac{\Delta x}{3}\left[g(a) + 4 \sum_{odd\, i} g(a + i\Delta x) + 2 \sum_{even\, i} g(a + i\Delta x) + g(b) \right]$$

Example

$$I = \int_2^3 \left(\int_2^3 (x^2 + y)dy \right) dx$$

$$I = \int_2^3 g(x)dx \qquad g(x) = \int_2^3 f(x,y)dy \qquad f(x,y) = x^2 + y$$

$$I = \int_2^3 g(x)dx \approx \frac{\Delta x}{2}\left[g(2)+2g(2.25)+2g(2.5) + 2g(2.75) + g(3) \right]$$

Example – Cont.

$$g(x) = \int_2^3 f(x,y)dy$$

$$\approx \frac{0.25}{2}\left[f(x,2) + 2f(x,\, 2.25) + 2f(x,2.5) + 2f(x,2.75) + f(x,3)\right]$$

$$g(2) = \int_2^3 f(2,y)dy$$

$$\approx \frac{0.25}{2}\left[f(2,2) + 2f(2,2.25) + 2f(2,2.5) + 2f(2,2.75) + f(2,3)\right]$$

$$= 6.5$$

Example – Cont.

$$g(2.25) = \int_2^3 f(2.25,y)dy \approx 7.56$$

$$g(2.5) = \int_2^3 f(2.5,y)dy \approx 8.75$$

$$g(2.75) = \int_2^3 f(2.75,y)dy \approx 10.06$$

$$g(3) = \int_2^3 f(3,y)dy \approx 11.5$$

$$I = \int_2^3 g(x)dx \approx \frac{0.25}{2}\left[6.5 + 2(7.56) + 2(8.75) + 2(10.06) + 11.5\right] = 8.437$$

Example – Cont.

```
Private Sub cmdGo_Click()

    Dim panels As Integer
    Dim a As Single, b As Single, c As Single, d As Single
    Dim Integral As Single

    panels = 4
    a = 2
    b = 3
    c = 2
    d = 3

    Integral = I(a, b, c, d, panels)
    picOutput.Print "Using ", panels, " I =  ", Integral

End Sub

Private Function I(a As Single, b As Single, c As Single, d As Single, n As Integer) As Single

    Dim sum As Single, x As Single, dx As Single
    Dim k As Integer
    sum = 0#
    dx = (b - a) / n
    sum = sum + g(a, c, d, n)
    For k = 1 To n - 1
        x = a + dx * (k)
        sum = sum + 2 * g(x, c, d, n)
    Next k
    sum = sum + g(d, c, d, n)
    I = (dx / 2) * sum
End Function
```

Example – Cont.

```
Private Function g(x As Single, c As Single, d As Single, n As Integer) As Single

    Dim sum As Single, y As Single, dy As Single
    Dim k As Integer
    sum = 0#
    dy = (d - c) / n
    sum = sum + f(x, c)
    For k = 1 To n - 1
        y = c + dy * (k)
        sum = sum + 2 * f(x, y)
    Next k
    sum = sum + f(x, d)
    g = (dy / 2) * sum
    picOutput.Print " g(", x, ") =  ", g
End Function

Private Function f(x As Single, y As Single) As Single

    f = x ^ 2 + y

End Function
```

Example – Cont.