Numerical Integration

• Purpose is to evaluate integrals
  – Impossible or very difficult to evaluate analytically
  – Functions available only at discrete points
  – Divide the interval into n equal subintervals (panels) \( \Delta x \) wide

\[
I = \int_{a}^{b} f(x) \, dx \\
\Delta x = \frac{b - a}{n}
\]
Trapezoid Rule

- Straight lines approximate the function between \(f(x_{i-1})\) and \(f(x_i)\), and \(f(x_i)\) and \(f(x_{i+1})\)

\[
\int_{x_{i-1}}^{x_i} f(x) \, dx = \frac{f_{i-1} + f_i}{2} \Delta x
\]

\[
\int_{x_i}^{x_{i+1}} f(x) \, dx = \frac{f_i + f_{i+1}}{2} \Delta x
\]

Trapezoid Rule

- For the two panels, we have

\[
\int_{x_{i-1}}^{x_{i+1}} f(x) \, dx = \int_{x_{i-1}}^{x_i} f(x) \, dx + \int_{x_i}^{x_{i+1}} f(x) \, dx
\]

\[
= \frac{\Delta x}{2} (f_{i-1} + f_i) + \frac{\Delta x}{2} (f_i + f_{i+1})
\]

\[
= \frac{\Delta x}{2} (f_{i-1} + 2f_i + f_{i+1})
\]

- Extending over entire interval

\[
\int_a^b f(x) \, dx = \frac{\Delta x}{2} (f_0 + 2f_1 + \cdots + 2f_{n-1} + f_n) = \frac{\Delta x}{2} \left( f_0 + 2 \sum_{i=1}^{n-1} f_i + f_n \right)
\]
Example - Trapezoid Rule

- Use trapezoidal rule with three panels to evaluate

\[ I = \int_0^{\pi} \sin(x) \, dx \]

\[ I = \frac{\pi}{3} \left[ f(0) + 2 f(\pi/3) + 2 f(2\pi/3) + f(\pi) \right] \]

\[ = \frac{\pi}{6} \left[ \sin(0) + 2 \sin(\pi/3) + 2 \sin(2\pi/3) + \sin(\pi) \right] \]

\[ = 0.523599 \left[ 0 + 2(0.866025) + 2(0.866025) + 0 \right] \]

\[ = 1.813799 \]

\[ I = \frac{\pi}{3} \left[ f(0) + 2 f(\pi/3) + 2 f(2\pi/3) + f(\pi) \right] \]

\[ = \frac{\pi}{3} \left[ \sin(0) + 2 \sin(\pi/3) + 2 \sin(2\pi/3) + \sin(\pi) \right] \]

\[ = 0.523599 \left[ 0 + 2(0.866025) + 2(0.866025) + 0 \right] \]

\[ = 1.813799 \]

Panel 1 2 3

Simpson’s 1/3 Rule

- Parabolic arcs are used to approximate the function between \( f(x_{i-1}) \) and \( f(x_i) \), and \( f(x_i) \) and \( f(x_{i+1}) \)

\[ I = \int_a^b f(x) \, dx = \int_a^b f_1(x) \, dx + \int_a^b f_2(x) \, dx \]
Simpson’s 1/3 Rule

\[ I = \int_{a}^{b} f(x) \, dx \approx \int_{a}^{b} f_2(x) \, dx \]

\[
f_2(x) = \frac{(x - x_i)(x - x_{i+1})}{(x_{i+1} - x_i)(x_{i+1} - x_{i+1})} f(x_{i+1}) + \frac{(x - x_{i-1})(x - x_i)}{(x_i - x_{i-1})(x_i - x_i)} f(x_i)
\]

\[ + \frac{(x - x_{i-1})(x - x_i)}{(x_{i+1} - x_{i-1})(x_{i+1} - x_i)} f(x_{i+1}) \]

**Simpson’s 1/3 Rule**

\[
\int_{x_{i-1}}^{x_{i+1}} f_2(x) \, dx = \int_{x_{i-1}}^{x_{i+1}} \left[ \frac{(x - x_i)(x - x_{i+1})}{(x_{i+1} - x_i)(x_{i+1} - x_{i+1})} f(x_{i+1}) + \frac{(x - x_{i-1})(x - x_i)}{(x_i - x_{i-1})(x_i - x_i)} f(x_i) \right. \\
\left. + \frac{(x - x_{i-1})(x - x_i)}{(x_{i+1} - x_{i-1})(x_{i+1} - x_i)} f(x_{i+1}) \right] \, dx \\
= \frac{\Delta x}{3} \left[ f(x_{i-1}) + 4 f(x_i) + f(x_{i+1}) \right]
\]
Simpson’s 1/3 Rule

Extending this over the entire interval

\[
\int_a^b f(x)\,dx = \frac{\Delta x}{3} (f_0 + 4f_1 + 2f_2 + \cdots + 4f_{n-1} + f_n)
\]

\[
= \frac{\Delta x}{3} (f_0 + 4 \sum_{i=1 \text{ odd}}^{n-1} f_i + 2 \sum_{i=2 \text{ even}}^{n-2} f_i + f_n)
\]

Example – Simpson’s 1/3 Rule

- Use Simpson’s rule with two panels to evaluate

\[
I = \int_0^\pi \sin(x)\,dx
\]

\[
I = \frac{\pi}{3} \left[ f(0) + 4f(\pi/4) + 2f(\pi/2) + f(\pi) \right]
\]

\[
= \frac{\pi}{12} \left[ \sin(0) + 4[\sin(\pi/4) + \sin(3\pi/4)] + 2\sin(\pi/2) + \sin(\pi) \right]
\]

\[
= \frac{\pi}{12} \left[ 0 + 4(0.7071 + 0.7071) + 0 + 0 \right]
\]

\[
= 0.2618[0 + 4(0.7071 + 0.7071) + 2(1.0) + 0]
\]

\[
= 2.004
\]
Example – Sailboat Mast

- Sailboat mast subject to distributed wind load. Total force resulting from wind load, \( F \), given by:

\[
F = \frac{30}{200} \int_{0}^{30} \left( \frac{z}{5+z} \right) e^{-2z/30} dz
\]

with line of action

\[
d = \frac{30}{200} \int_{0}^{30} \left( \frac{z}{5+z} \right) e^{-2z/30} dz
\]

Find the tensile force \( T \) in the left support cable.

Example – Trapezoid Rule

Private Sub cmdGo_Click()

Dim panels As Integer
Dim a As Single, b As Single, Integral As Single

panels = 1000
a = 0.5
b = 50#

Integral = f(a, b, panels)
picOutput.Print "Using ", panels, " the integral = ", Integral

End Sub

Private Function f(a As Single, b As Single, n As Integer) As Single

Dim sum As Single, x As Single, dx As Single
Dim k As Integer
sum = 0
dx = (b - a) / n
For k = 1 To n
x = a + dx * (k - 1)
sum = sum + f(x)
Next
f = 0.5 * dx * (f(a) + f(b) + 2 * sum)

End Function
Example – Trapezoid Rule

- Total Force, $F$, Line of Action, $d$

\[
F = 3000 \times (x / |5x + x|) \times \exp(-2 \times x / 30)
\]

\[
E = 3000 \times x \times (|5x + x|) \times \exp(-2 \times x / 30)
\]

\[
d = 19326.89 / 1480.57 = 13.05 \text{ ft}
\]

Example – Sailboat Mast

- Free body diagram

\[
\sum F_H = 0 = F - T \sin \theta - H
\]

\[
\sum F_V = 0 = V - T \cos \theta
\]

\[
\sum M_0 = 0 = 3V - Fd
\]

\[
V = \frac{Fd}{3} = \frac{1480.6 \times 13.05}{3} = 6440.6 \text{ lb}
\]

\[
T = \frac{V}{\cos \theta} = \frac{6440.6}{0.995} = 6473 \text{ lb}
\]

\[
H = F - T \sin \theta = 1480.6 - 6473 \times 0.0995 = 836.54 \text{ lb}
\]
Multiple Integrals

\[ I = \int_a^b \left( \int_{c(x)}^{d(x)} f(x,y) \, dy \right) \, dx = \int_a^b g(x) \, dx \]

\[ g(x) = \frac{d(x)}{c(x)} \int_{c(x)}^{d(x)} f(x,y) \, dy \]

Sometimes \( c(x) \) and \( d(x) \) are constants, \( c \) and \( d \)

\[ I = \int_a^b g(x) \, dx = \frac{\Delta x}{3} \left[ g(a) + 4 \sum_{\text{odd } i} g(a + i\Delta x) + 2 \sum_{\text{even } i} g(a + i\Delta x) + g(b) \right] \]

Example

\[ I = \int_2^3 \left( \int_2^3 (x^2 + y) \, dy \right) \, dx \]

\[ I = \int_2^3 g(x) \, dx \quad g(x) = \frac{3}{2} \int_2^3 f(x,y) \, dy \quad f(x,y) = x^2 + y \]

\[ I = \int_2^3 g(x) \, dx = \frac{\Delta x}{2} \left[ g(2) + 2 \frac{2g(2.25) + 2g(2.5) + 2g(2.75) + g(3)}{3} \right] \]
Example – Cont.

\[ g(x) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(x,y) \, dy \]
\[ = \frac{0.25}{2} \left[ f(x, 2) + 2f(x, 2.25) + 2f(x, 2.5) + 2f(x, 2.75) + f(x, 3) \right] \]

\[ g(2) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(2, y) \, dy \]
\[ = \frac{0.25}{2} \left[ f(2, 2) + 2f(2, 2.25) + 2f(2, 2.5) + 2f(2, 2.75) + f(2, 3) \right] \]
\[ = 6.5 \]

Example – Cont.

\[ g(2.25) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(2.25, y) \, dy = 7.56 \]
\[ g(2.5) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(2.5, y) \, dy = 8.75 \]
\[ g(2.75) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(2.75, y) \, dy = 10.06 \]
\[ g(3) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(3, y) \, dy = 11.5 \]

\[ I = \int_{\frac{1}{2}}^{\frac{3}{2}} g(x) \, dx = \frac{0.25}{2} \left[ 6.5 + 2(7.56) + 2(8.75) + 2(10.06) + 11.5 \right] = 8.437 \]
Example – Cont.

Private Sub cmdStart_Click()
  Dim panels As Integer
  Dim a As Single, b As Single, c As Single, d As Single
  Dim Integral As Single

  panels = 4
  a = 1
  b = 2
  c = 3
  d = 3

  Integral = f(a, b, c, d, panels)
  picOutput.Print "Integral = ", Integral
End Sub

Private Function f(x As Single, y As Single, a As Single, b As Single, c As Single, d As Integer) As Single
  Dim sum As Single
  Dim k As Integer
  Dim g As Single

  sum = 0
  g = (dy / 2) * sum
  sum = sum + 2 * f(x, y)
  g = g + 2 * sum

  Next k
  g = d + g
  g = g + 2 * sum
  picOutput.Print "Integral = ", g
End Function

Example – Cont.

Private Function g(x As Single, c As Single, d As Single, n As Integer) As Single
  Dim sum As Integer
  Dim x As Single, y As Single
  Dim k As Integer
  Dim m As Integer
  Dim n As Integer
  Dim dy As Single

  sum = 0
  dy = (d - c) / n
  sum = sum + f(x, c)
  For k = 1 To n - 1
    y = c + dy * k
    sum = sum + 2 * f(x, y)
  Next k
  sum = sum + 2 * f(x, c)
  m = m + 2
  picOutput.Print ",
End Function

Private Function f(x As Single, y As Single) As Single
  f = x ^ 2 + y
End Function
Example – Cont.

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