Curve Fitting

CE 311 K - Introduction to Computer Methods
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Nonlinear Regression

- Minimize the residual between the data points and the curve -- least-squares regression
  - Linear
    - $y_i = a_0 + a_1 x_i$
  - Quadratic
    - $y_i = a_0 + a_1 x_i + a_2 x_i^2$
  - ...
  - Exponential (base $e$)
    - $y_i = a e^{bx_i}$
  - Power (base $x$)
    - $y_i = a x_i^b$
  - Saturation-Growth
    - $y_i = a \frac{x_i}{b + x_i}$
Exponential Relationship

- If the relationship is an exponential function
  \[ y_i = ae^{bx_i} \]
- To make it linear, take logarithm of both side
  \[ \ln(y_i) = \ln(a) + bx_i \rightarrow Y_i = A + bx_i \]
- Now it’s a linear relation between \( Y (=\ln(y)) \) and \( x \)
- Need to estimate the values of \( A (=\ln(a)) \) and \( b \)

Power Relationship

- If the relationship is a power function
  \[ y_i = ax_i^b \]
- To make it linear, take logarithm of both side
  \[ \ln(y_i) = \ln(a) + b\ln(x_i) \rightarrow Y_i = A + bX_i \]
- Now it’s linear between \( Y (=\ln(y)) \) and \( X (=\ln(x)) \)
- Need to estimate the values of \( A (=\ln(a)) \) and \( b \)
Saturation-Growth Relationship

• If the relationship is a saturation-growth function
  \[ y_i = \frac{ax_i}{b + x_i} \]
  
• To make it linear, invert the equation
  \[ \frac{1}{y} = \frac{b}{x} + \frac{1}{a} \]
  
  \[ Yi = A + BX_i \]

• Now it’s linear between \( Y = 1/y \) and \( X = 1/x \)

• Need to estimate the values of \( A = 1/a \) and \( B = b/a \)

Some Examples

• Quadratic curve \( y = a_0 + a_1x + a_2x^2 \)
  
  \( Q = a_0 + a_1H + a_2H^2 \)

• Flow rating curve:
  \- Q = measured discharge,
  \- H = stage (height) of water behind outlet

• Power curve \( y = ax^b \)

  \( c = aq^b \)

  \- Sediment transport:
    \- c = concentration of suspended sediment
    \- q = river discharge

  \- Carbon adsorption:
    \- q = mass of pollutant sorbed per unit mass of carbon,
    \- C = concentration of pollutant in solution
Example – Power Function

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>Log(x)</th>
<th>Log(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>2.1</td>
<td>0.18</td>
<td>0.74</td>
</tr>
<tr>
<td>2.8</td>
<td>11.5</td>
<td>1.03</td>
<td>2.44</td>
</tr>
<tr>
<td>4.3</td>
<td>28.1</td>
<td>1.46</td>
<td>3.34</td>
</tr>
<tr>
<td>5.4</td>
<td>41.9</td>
<td>1.69</td>
<td>3.74</td>
</tr>
<tr>
<td>6.8</td>
<td>72.3</td>
<td>1.92</td>
<td>4.28</td>
</tr>
<tr>
<td>7.9</td>
<td>91.4</td>
<td>2.07</td>
<td>4.52</td>
</tr>
</tbody>
</table>

Example – Power Function

- Using the log’s, not the original x’s and y’s

\[
\begin{bmatrix}
\frac{n}{\sum_{i=1}^{n} X_i} \\
\frac{n}{\sum_{i=1}^{n} X_i^2}
\end{bmatrix} \begin{bmatrix}
a \\
b
\end{bmatrix} = \begin{bmatrix}
\frac{5}{\sum_{i=1}^{n} Y_i} \\
\frac{5}{\sum_{i=1}^{n} X_i Y_i}
\end{bmatrix}
\]

\[
\begin{bmatrix}
6 \\
8.34
\end{bmatrix} \begin{bmatrix}
a \\
b
\end{bmatrix} = \begin{bmatrix}
19.1 \\
31.4
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{5}{\sum_{i=1}^{n} X_i} \\
\frac{5}{\sum_{i=1}^{n} X_i^2}
\end{bmatrix} = \begin{bmatrix}
8.34 \\
14.0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{5}{\sum_{i=1}^{n} Y_i} \\
\frac{5}{\sum_{i=1}^{n} X_i Y_i}
\end{bmatrix} = \begin{bmatrix}
19.1 \\
31.4
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{5}{\sum_{i=1}^{n} (x_i \ln(y_i))}
\end{bmatrix} = 31.4
\]