Solving Nonlinear Equations

CE 311 K - Introduction to Computer Methods

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Nonlinear Systems

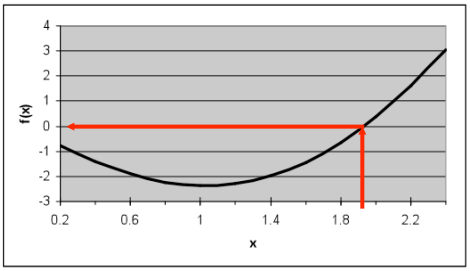
- Linear system behavior is the sum of its parts
- Nonlinear system behavior is not
- Linearity allows for easier computation of results
- Nonlinear systems are not so easy to solve
- Some nonlinear systems are
  - exactly solvable or integrable,

  others are known to be
  - chaotic, and thus have no simple or closed form solution

Nonlinear Equations

$$f(x) = 0$$

- Example $f(x) = x^2 - 4\sin(x) = 0$



x = 1.9 is a root

How do we find it?

Finding Roots of Equations (Solving Nonlinear Equations)

- Three Methods
  - Fixed Point Iteration
  - Bisection
  - Newton

Fixed Point Iteration

- Find roots by rearranging equation
- Example

$$f(x) = \sin(\sqrt{x}) - x = 0$$ Original equation

$$x = \sin(\sqrt{x})$$

$$g(x) = \sin(\sqrt{x})$$ Rearranged equation

- Given an initial guess of the root, iterate to improve it

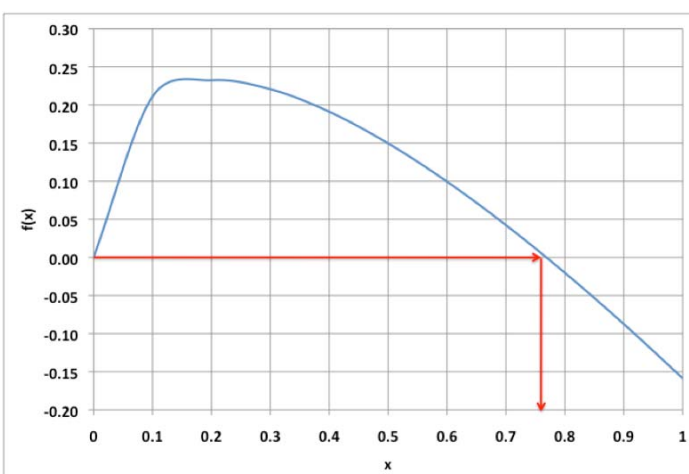
Initial Guess

Improved value

$$x_{i+1} = g(x_i) = \sin(\sqrt{x_i})$$

Example

$$f(x) = \sin(\sqrt{x}) - x = 0$$



Fixed Point Iteration

- Example

$$f(x) = \sin(\sqrt{x}) - x = 0$$ Original equation

$$x = \sin(\sqrt{x})$$

$$g(x) = \sin(\sqrt{x})$$ Rearranged equation

- Given an initial guess of the root, iterate to improve it

Initial Guess

Improved value

$$x_{i+1} = g(x_i) = \sin(\sqrt{x_i})$$

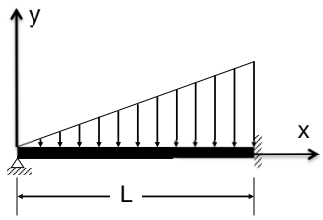
Example – Fixed Point Iteration

$$x_{i+1} = g(x_i) = \sin(\sqrt{x_i})$$

x_i	x_{i+1}	error
1.00000	0.84147	2.E-01
0.84147	0.79397	6.E-02
0.79397	0.77773	2.E-02
0.77773	0.77194	8.E-03
0.77194	0.76985	3.E-03
0.76985	0.76909	1.E-03
0.76909	0.76881	4.E-04
0.76881	0.76871	1.E-04
0.76871	0.76867	5.E-05
0.76867	0.76866	2.E-05
0.76866	0.76865	6.E-06
0.76865	0.76865	2.E-06
0.76865	0.76865	9.E-07

Example – Beam Deflection

- Uniform beam with linearly increasing load

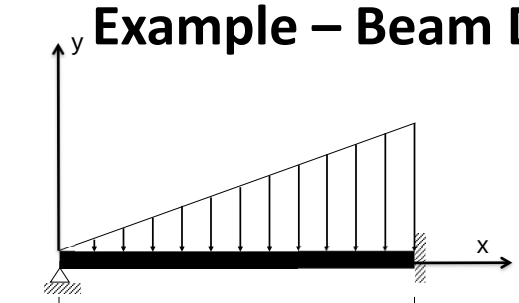


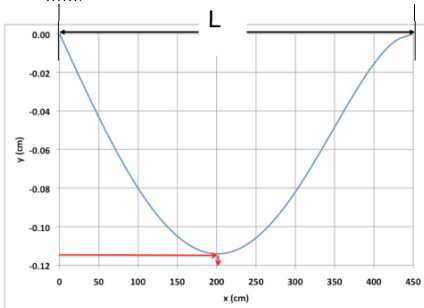
Given:
w_0 = 1.75 kN/cm
E = 50,000 kN/cm^2;
I = 30,000 cm^4; and
L = 450 cm;

$$y = \frac{w_0}{120EIL}\left(-x^5 + 2L^2x^3 - L^4x\right)$$

Find: Point (x) of maximum deflection

Example – Beam Deflection



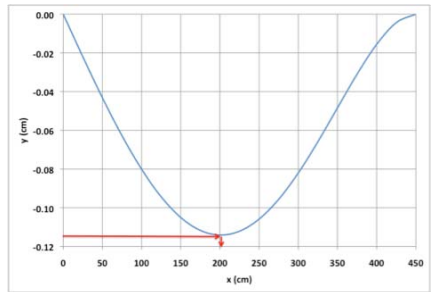


x (cm)	y (cm)
0	0.0000
25	-0.0220
50	-0.0432
75	-0.0628
100	-0.0801
125	-0.0943
150	-0.1050
175	-0.1117
200	-0.1141
225	-0.1121
250	-0.1059
275	-0.0956
300	-0.0820
325	-0.0659
350	-0.0484
375	-0.0310
400	-0.0156
425	-0.0044
450	0.0000

$$y = \frac{w_0}{120EIL}\left(-x^5 + 2L^2x^3 - L^4x\right)$$

Example – Beam Deflection

- Maximum deflection at point *x* where *dy*/*dx* = 0



$$y = \frac{w_0}{120EIL}\left(-x^5 + 2L^2x^3 - L^4x\right)$$

$$\frac{dy}{dx} = \frac{w_0}{120EIL}\left(-5x^4 + 6L^2x^2 - L^4\right) = 0$$

$$f(x) = -5x^4 + 6L^2x^2 - L^4 = 0$$

Example – Beam Deflection

- Rearrange & use Fixed Point iteration

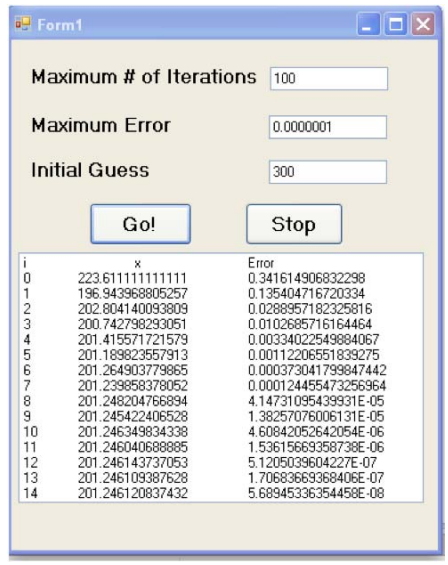
$$f(x) = -5x^4 + 6L^2x^2 - L^4 = 0$$

$$g(x) = \frac{5x^4 + L^4}{6L^2x}$$

$$x_{i+1} = g(x_i) = \frac{-5x_i^4 - L^4}{6L^2x_i}$$

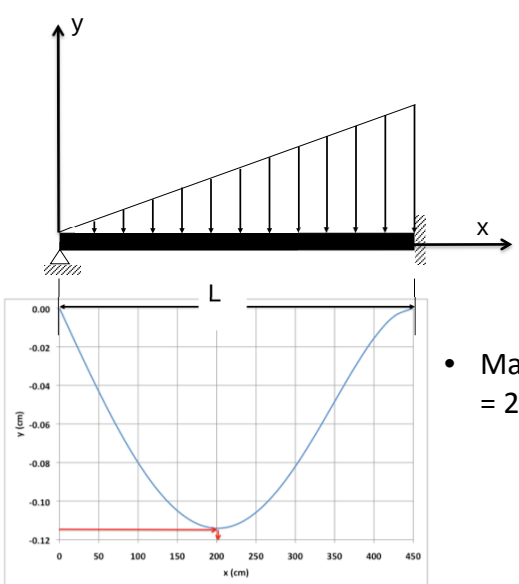
Example – Beam Deflection

```
Private Sub Button1_Click(ByVal sender As System.Objec
    Dim xn, xo, err, e_max As Double
    Dim i, n As Integer

    n = CInt(TextBox1.Text)
    e_max = CDbl(TextBox2.Text)
    xo = CDbl(TextBox3.Text)

    i = 0
    err = 100000000.0
    ListBox1.Items.Add("i" & vbTab & vbTab & _
                       "x" & vbTab & vbTab & "Error")

    Do Until (i > n) Or (err < e_max)
        xn = g(xo)
        err = Math.Abs((xn - xo) / xn)
        ListBox1.Items.Add(i & vbTab & _
                           xn & vbTab & vbTab & err)
        xo = xn
        i = i + 1
    Loop

End Sub

Function g(ByVal x As Double) As Double

    g = (5 * x ^ 4 + 450 ^ 4) / (6 * 450 ^ 2 * x)

End Function
```



i	x	Error
0	223.611111111111	0.341614906832298
1	196.943968805257	0.135404716720334
2	202.804140093809	0.0288957182325816
3	200.742798293051	0.0102685716164464
4	201.415571721579	0.00334022549884067
5	201.189823557913	0.00112206551839275
6	201.264903779865	0.000373041799847442
7	201.239858378052	0.000124455473256964
8	201.248204766894	4.14731095439931E-05
9	201.245422406528	1.38257076006131E-05
10	201.246349834338	4.60842052642054E-06
11	201.246040688885	1.53615669358738E-06
12	201.246143737053	5.1205039604227E-07
13	201.246109387628	1.70683669368406E-07
14	201.246120837432	5.68945336354458E-08

Example – Beam Deflection



- Maximum deflection at: x = 201 cm, y = -0.11 cm