Solving Nonlinear Equations

CE 311 K - Introduction to Computer Methods

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Nonlinear Systems

- Linear system behavior is the sum of its parts
- Nonlinear system behavior is not
- Linearity allows for easier computation of results
- Nonlinear systems are not so easy to solve
- Some nonlinear systems are
  - exactly solvable or integrable,
- others are known to be
  - chaotic, and thus have no simple or closed form solution
Nonlinear Equations

\[ f(x) = 0 \]

• Example  \[ f(x) = x^2 - 4\sin(x) = 0 \]

\[ x = 1.9 \] is a root

How do we find it?

Finding Roots of Equations
(Solving Nonlinear Equations)

• Three Methods
  – Fixed Point Iteration
  – Bisection
  – Newton
Fixed Point Iteration

• Find roots by rearranging equation

• Example
  \[ f(x) = \sin(\sqrt{x}) - x = 0 \]
  \[ x = \sin(\sqrt{x}) \]
  \[ g(x) = \sin(\sqrt{x}) \]

  Rearranged equation

• Given an initial guess of the root, iterate to improve it

  \[ x_{i+1} = g(x_i) = \sin(\sqrt{x_i}) \]
  
Example

\[ f(x) = \sin(\sqrt{x}) - x = 0 \]
Fixed Point Iteration

- Example
  \[ f(x) = \sin(\sqrt{x}) - x = 0 \]
  \[ x = \sin(\sqrt{x}) \]
  \[ g(x) = \sin(\sqrt{x}) \]

- Given an initial guess of the root, iterate to improve it

\[ x_{i+1} = g(x_i) = \sin(\sqrt{x_i}) \]

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Example – Fixed Point Iteration

\[ x_{i+1} = g(x_i) = \sin(\sqrt{x_i}) \]

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( x_{i+1} )</th>
<th>( \text{error} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000</td>
<td>0.84147</td>
<td>2.E-01</td>
</tr>
<tr>
<td>0.84147</td>
<td>0.79397</td>
<td>6.E-02</td>
</tr>
<tr>
<td>0.79397</td>
<td>0.77773</td>
<td>2.E-02</td>
</tr>
<tr>
<td>0.77773</td>
<td>0.77194</td>
<td>8.E-03</td>
</tr>
<tr>
<td>0.77194</td>
<td>0.76985</td>
<td>3.E-03</td>
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<tr>
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<td>0.76909</td>
<td>1.E-03</td>
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<tr>
<td>0.76909</td>
<td>0.76881</td>
<td>4.E-04</td>
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<tr>
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<td>0.76871</td>
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<tr>
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</tr>
<tr>
<td>0.76865</td>
<td>0.76865</td>
<td>9.E-07</td>
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</table>
Example – Beam Deflection

• Uniform beam with linearly increasing load

\[ y = \frac{w_0}{120EIL} \left( -x^5 + 2L^2x^3 - L^4x \right) \]

Find: Point (x) of maximum deflection

Given:
\[ w_0 = 1.75 \text{ kN/cm} \]
\[ E = 50,000 \text{ kN/cm}^2; \]
\[ I = 30,000 \text{ cm}^4; \] and
\[ L = 450 \text{ cm}; \]

<table>
<thead>
<tr>
<th>x (cm)</th>
<th>y (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0000</td>
</tr>
<tr>
<td>25</td>
<td>-0.0220</td>
</tr>
<tr>
<td>50</td>
<td>-0.0432</td>
</tr>
<tr>
<td>75</td>
<td>-0.0628</td>
</tr>
<tr>
<td>100</td>
<td>-0.0801</td>
</tr>
<tr>
<td>125</td>
<td>-0.0943</td>
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<tr>
<td>150</td>
<td>-0.1050</td>
</tr>
<tr>
<td>175</td>
<td>-0.1117</td>
</tr>
<tr>
<td>200</td>
<td>-0.1141</td>
</tr>
<tr>
<td>225</td>
<td>-0.1121</td>
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<tr>
<td>250</td>
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<tr>
<td>275</td>
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<tr>
<td>300</td>
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<tr>
<td>325</td>
<td>-0.0659</td>
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<tr>
<td>425</td>
<td>-0.0044</td>
</tr>
<tr>
<td>450</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
Example – Beam Deflection

• Maximum deflection at point x where \( dy/dx = 0 \)

\[
y = \frac{w_0}{120EI} \left( -x^5 + 2L^2x^3 - L^4x \right)
\]

\[
\frac{dy}{dx} = -\frac{w_0}{120EI} \left( -5x^4 + 6L^2x^2 - L^4 \right) = 0
\]

\[
f(x) = -5x^4 + 6L^2x^2 - L^4 = 0
\]

Example – Beam Deflection

• Rearrange & use Fixed Point iteration

\[
f(x) = -5x^4 + 6L^2x^2 - L^4 = 0
\]

\[
g(x) = \frac{5x^4 + L^4}{6L^2x}
\]

\[
x_{i+1} = g(x_i) = \frac{-5x_i^4 - L^4}{6L^2x_i}
\]
Example – Beam Deflection

Private Sub Button1_Click(ByVal sender As System.Object, ByVal e As System.EventArgs) Handles Button1.Click
    Dim x0, y0, err, a_max As Double
    Dim i, n As Integer

    a = CStr(Text1.Text)
    a_max = CStr(Text2.Text)
    r0 = CStr(Text3.Text)

    L = 0
    err = 10000000.0
    List1.Items.Add("i" & vbCrLf & "x = " & vbCrLf & "Error")

    Do Until (i > n) Or (err < a_max)
        x0 = r0
        y0 = g(x0, a)
        r0 = x0 + a_max * x0
        y0 = g(r0, a)
        err = Math.Abs((y0 - y0) / y0)
        List1.Items.Add(i & vbCrLf & x & vbCrLf & y & vbCrLf & "Error")
        i = i + 1
    Loop
End Sub

Function g(ByVal x As Double) As Double
    a = (5 * x ^ 4 + 450 * x) / (4 * 450 ^ 2 * x)
End Function

Example – Beam Deflection

- Maximum deflection at: x = 201 cm, y = 0.11 cm