Solution of Linear Equations
1. Direct Methods

CE 311 K - Introduction to Computer Methods

Daene C. McKinney

Systems of Linear Equations

- Consider the linear system $\boldsymbol{Ax} = \boldsymbol{b}$
- where $\boldsymbol{A}$ is an (n x n) matrix, $\boldsymbol{x}$ is the vector of (n) unknown solution values, and $\boldsymbol{b}$ is a column vector of constants

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Solution of a Linear System

- A formal way to obtain a solution using matrix algebra is to multiply each side of the equation by the inverse of ***A*** to yield

$$A^{-1}Ax = A^{-1}b$$

$$x = A^{-1}b$$

Gauss Elimination

- ***Forward Elimination*** - multiple of one equation is subtracted from another to eliminate unknowns
- ***Back Substitution*** – last equation yields unknown, substituting back into the other equations yields the rest

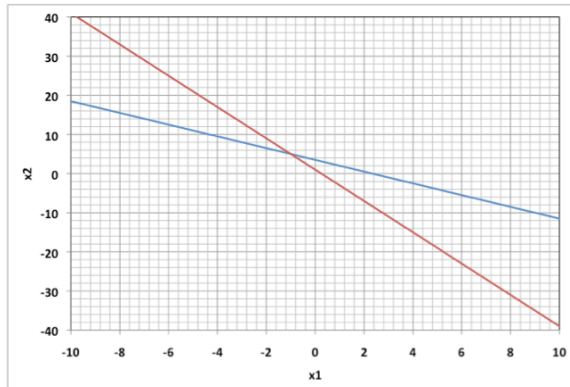
Example

- Consider the system

$$3x_1 + 2x_2 = 7$$
$$4x_1 + x_2 = 1$$

$$x_2 = \frac{-3}{2}x_1 + \frac{7}{2}$$
$$x_2 = -4x_1 + 1$$

$$x_2 = 5$$
$$x_1 = -1$$



Gauss Elimination – Forward Elimination

- Consider the two equations from the previous example
- Divide the first equation by 3, multiply it by 4 and subtract it from the second equation, yielding the new system of equations

Use First Row

-4/3 * $\longrightarrow 3x_1 + 2x_2 = 7$

Add $\longrightarrow 4x_1 + x_2 = 1$

Result

$$3x_1 + 2x_2 = 7$$
$$-\frac{5}{3}x_2 = -\frac{25}{3}$$

Gauss Elimination – Back Substitution

- Solve the second equation for x2
- Substitute back into the first equation, solve for x_1

$$x_2 = 5$$

$$x_1 = \frac{7 - 2(5)}{3} = -1$$

Gauss Elimination

- n equations in n unknowns

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

- x is unchanged if we:
  - Multiply or divide any equation by a constant
  - Replace any equation by the sum of that equation and any other equation

Forward Elimination – Step 1

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

a_{21}/a_{11}

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
0 + a'_{22}x_2 + a_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Forward Elimination – Step 2 - n

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
0 + a'_{22}x_2 + a_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

a_{31}/a_{11}

a_{n1}/a_{11}

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a'_{32}x_2 + a'_{33}x_3 + \cdots + a'_{3n}x_n &= b'_3 \\
&\vdots \\
a'_{n2}x_2 + a'_{n3}x_3 + \cdots + a'_{nn}x_n &= b'_n
\end{aligned}
$$

Forward Elimination – Last Step

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a'_{32}x_2 + a'_{33}x_3 + \cdots + a'_{3n}x_n &= b'_3 \\
&\vdots \\
a'_{n2}x_2 + a'_{n3}x_3 + \cdots + a'_{nn}x_n &= b'_n
\end{aligned}
$$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a^{\cdots}_{nn}x_n &= b^{\cdots}_n
\end{aligned}
$$

Back Substitution – Step 1

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a^{\cdots}_{nn}x_n &= b^{\cdots}_n
\end{aligned}
$$

$$x_n = \frac{b^{\cdots}_n}{a^{\cdots}_{nn}}$$

Back Substitution – Steps 2 - n

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a^{\cdots}_{nn}x_n &= b^{\cdots}_n
\end{aligned}
$$

$$x_n = \frac{b^{\cdots}_n}{a^{\cdots}_n}$$

$$a^{\cdots}_{n-1,n-1}x_{n-1} + a^{\cdots}_{n-1,n}x_n = b^{\cdots}_{n-1}$$

$$x_{n-1} = \frac{b^{\cdots}_{n-1} - a^{\cdots}_{n-1,n}x_n}{a''^{\cdots}_{n-1,n-1}}$$

Finally

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)}{a_{11}} = \frac{b_1 - (\sum_{k=2}^{n} a_{1k}x_k)}{a_{11}}$$

Solution Methods

- Gauss Elimination – Subject to roundoff errors and ill conditioning
- Iterative methods -- Alternative to elimination method
- Take initial guess of solution and then iterate to obtain improved estimates of the solution
- Jacobi and Gauss-Seidel methods
- Work well for large sets of equations

Pivoting

- Consider the system

$$\begin{aligned} x_2 + x_3 &= 1 \\ x_1 + x_3 &= 1 \\ x_1 + x_2 &= 1 \end{aligned}$$

- In Gauss Elimination, we use the kth equation to eliminate x_k from equations k+1, k+2, …, n
- This is possible only when the coefficient $a_{kk} \neq 0$
- We could exchange the first and second equations

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + x_3 &= 1 \\ x_1 + x_2 &= 1 \end{aligned}$$

Pivoting

- From which we get:

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + x_3 &= 1 \\ x_1 + x_2 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + x_3 &= 1 \\ x_2 - x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + x_3 &= 1 \\ -2x_3 &= -1 \end{aligned}$$

$$x_1 = x_2 = x_3 = \frac{1}{2}$$

Roundoff Errors in Gauss Elimination

- Consider
$$\begin{aligned} 0.0003x_1 \quad +1.566x_2 &= 1.569 \\ 0.3454x_1 \quad -2.436x_2 &= 1.018 \end{aligned}$$
- The solution is $x_1 = 10 \quad x_2 = 1$
- However, if we only have 4-decimal arithmetic and solve with G-Elimination, we have

$$\begin{aligned} 0.0003x_1 \quad +1.566x_2 &= 1.569 \\ 0.3454x_1 \quad -2.436x_2 &= 1.018 \end{aligned}$$

$$a_{21}/a_{11} = \frac{0.3454}{0.0003} = 1151$$

$$\begin{aligned} 0.0003x_1 \quad +1.566x_2 &= 1.569 \\ -1804x_2 &= 1805 \end{aligned}$$

$$x_2 = \frac{-1805}{-1804} = 1.001$$

$$x_1 = \frac{1.569 - (1.566)(1.001)}{0.0003} = 3.333$$

Roundoff

- The problem is that $|a_{11}| << |a_{12}|$
- Thus a small error due to roundoff in x_2 leads to a large error in x_1
- What happens of we reverse the order of the eqautions ad solve? Try it!!

$$\begin{aligned} 0.3454x_1 \quad -2.436x_2 &= 1.018 \\ 0.0003x_1 \quad +1.566x_2 &= 1.569 \end{aligned}$$