Solution of Linear Equations

1. Direct Methods

**CE 311 K - Introduction to Computer Methods**

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Systems of Linear Equations

- Consider the linear system $Ax = b$
- where $A$ is an $(n \times n)$ matrix, $x$ is the vector of $(n)$ unknown solution values, and $b$ is a column vector of constants

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix} = 
\begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_n
\end{pmatrix}
\]

\[
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\]
Solution of a Linear System

- A formal way to obtain a solution using matrix algebra is to multiply each side of the equation by the inverse of $A$ to yield

$$A^{-1}Ax = A^{-1}b$$
$$x = A^{-1}b$$

Gauss Elimination

- *Forward Elimination* - multiple of one equation is subtracted from another to eliminate unknowns

- *Back Substitution* – last equation yields unknown, substituting back into the other equations yields the rest
Example

- Consider the system

\[ \begin{align*}
3x_1 + 2x_2 &= 7 \\
4x_1 + x_2 &= 1 \\
x_2 &= \frac{-3}{2}x_1 + \frac{7}{2} \\
x_2 &= -4x_1 + 1 \\
x_2 &= 5 \\
x_1 &= -1
\end{align*} \]

Gauss Elimination – Forward Elimination

- Consider the two equations from the previous example

- Divide the first equation by 3, multiply it by 4 and subtract it from the second equation, yielding the new system of equations

\[ \begin{align*}
\text{Use First Row} & \quad 3x_1 + 2x_2 = 7 \\
\text{Result} & \quad 3x_1 + 2x_2 = 7 \\
\text{-4/3} \cdot & \quad x_1 \quad \rightarrow 3x_1 + 2x_2 = 7 \\
\text{Add} & \quad 4x_1 + x_2 = 1 \\
\quad & \quad \frac{-5}{3}x_2 = \frac{-25}{3}
\end{align*} \]
Gauss Elimination – Back Substitution

• Solve the second equation for $x_2$
• Substitute back into the first equation, solve for $x_1$

\[ x_2 = 5 \]
\[ x_1 = \frac{7 - 2(5)}{3} = -1 \]

Gauss Elimination

• $n$ equations in $n$ unknowns

\[ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \]
\[ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \]
\[ \vdots \]
\[ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \]

• $x$ is unchanged if we:
  – Multiply or divide any equation by a constant
  – Replace any equation by the sum of that equation and any other equation
Forward Elimination – Step 1

\[ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \]
\[ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \]
\[ \vdots \]
\[ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \]

Forward Elimination – Step 2 - n

\[ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \]
\[ 0 + a'_{22}x_2 + a_{23}x_3 + \cdots + a'_{2n}x_n = b'_2 \]
\[ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \]
\[ \vdots \]
\[ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \]
Forward Elimination – Last Step

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
    a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
    a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
    \vdots \\
    a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\]

Back Substitution – Step 1

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
    a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
    a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
    \vdots \\
    a_{nn}x_n &= b_n
\end{align*}
\]

\[
x_n = \frac{b_n}{a_{nn}}
\]
Back Substitution – Steps 2 - n

\[ x_n = \frac{b_n}{a_n} \]
\[ a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1} \]

\[ x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} \]

Finally

\[ x_j = \frac{b_j - (a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)}{a_{11}} = \frac{b_j - (\sum_{k=2}^{n} a_{jk}x_k)}{a_{11}} \]

Solution Methods

- Gauss Elimination – Subject to roundoff errors and ill conditioning
- Iterative methods -- Alternative to elimination method
- Take initial guess of solution and then iterate to obtain improved estimates of the solution
- Jacobi and Gauss-Seidel methods
- Work well for large sets of equations
Pivoting

• Consider the system

\[
\begin{align*}
  x_2 + x_3 &= 1 \\
  x_1 + x_3 &= 1 \\
  x_1 + x_2 &= 1
\end{align*}
\]

• In Gauss Elimination, we use the \( k \)th equation to eliminate \( x_k \) from equations \( k+1, k+2, \ldots, n \)
• This is possible only when the coefficient \( a_{kk} \neq 0 \)
• We could exchange the first and second equations

\[
\begin{align*}
  x_1 + x_3 &= 1 \\
  x_2 + x_3 &= 1 \\
  x_1 + x_2 &= 1
\end{align*}
\]

Pivoting

• From which we get:

\[
\begin{align*}
  x_1 + x_3 &= 1 \\
  x_2 + x_3 &= 1 \\
  x_1 + x_2 &= 1 \\
  x_1 + x_3 &= 1 \\
  x_2 + x_3 &= 1 \\
  x_2 - x_3 &= 0 \\
  x_1 + x_3 &= 1 \\
  x_2 + x_3 &= 1 \\
  -2x_3 &= -1
\end{align*}
\]

\[
\begin{align*}
  x_1 &= x_2 = x_3 = \frac{1}{2}
\end{align*}
\]
Roundoff Errors in Gauss Elimination

- Consider \[0.0003x_1 + 1.566x_2 = 1.569\]
  \[0.3454x_1 - 2.436x_2 = 1.018\]
- The solution is \[x_1 = 10\quad x_2 = 1\]
- However, if we only have 4-decimal arithmetic and solve with G-Elimination, we have
  \[
  \begin{align*}
  0.0003x_1 &+ 1.566x_2 = 1.569 & \quad a_{21}/a_{11} = \frac{0.3454}{0.0003} \approx 1151 \\
  0.3454x_1 &- 2.436x_2 = 1.018 & \\
  0.0003x_1 &+ 1.566x_2 = 1.569 & \\
  -1804x_2 & = 1805 & \\
  x_2 &= \frac{-1805}{-1804} = 1.001 \\
  x_1 &= \frac{1.569 - (1.566)(1.001)}{0.0003} = 3.333
  \end{align*}
  \]

Roundoff

- The problem is that \(|a_{11}| \ll |a_{12}|\)
- Thus a small error due to roundoff in \(x_2\) leads to a large error in \(x_1\)
- What happens if we reverse the order of the equations and solve? Try it!!
  \[
  \begin{align*}
  0.3454x_1 &- 2.436x_2 = 1.018 \\
  0.0003x_1 &+ 1.566x_2 = 1.569
  \end{align*}
  \]