

Matrices

*CE 311 K - Introduction to Computer
Methods*

Daene C. McKinney

Introduction

- Matrices
- Matrix Arithmetic
 - Addition
 - Multiplication
- Diagonal Matrices
 - Identity matrix
- Matrix Inverse

Matrix

- Matrix - a rectangular array of numbers arranged into m rows and n columns:

$$\begin{array}{c} \text{Rows,} \\ i = 1, \dots, m \end{array} \left\{ \begin{array}{cccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array} \right.$$

Columns,
 $j = 1, \dots, n$

Matrices

- Variety of engineering problems lead to the need to solve systems of linear equations

$$Ax = b$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

matrix
column vectors

Row and Column Matrices (vectors)

- Row matrix (or row vector) is a matrix with one row

$$r = (r_1 \ r_2 \ r_3 \ \cdots \ r_n)$$

- Column vector is a matrix with one column

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

Square Matrix

- When the row and column dimensions of a matrix are equal ($m = n$) then the matrix is called square

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Matrix Transpose

- The transpose of the $(m \times n)$ matrix A is the $(n \times m)$ matrix formed by interchanging the rows and columns such that row i becomes column i of the transposed matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & & \ddots & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Matrix Equality

- Two $(m \times n)$ matrices A and B are equal if and only if each of their elements are equal. That is

$$A = B$$

if and only if

$$a_{ij} = b_{ij} \text{ for } i = 1, \dots, m; j = 1, \dots, n$$

Vector Addition

- The sum of two ($m \times 1$) column vectors \mathbf{a} and \mathbf{b} is

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_m + b_m \end{pmatrix}$$

Matrix Addition

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & & a_{2n} + b_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

Matrix Multiplication

- The product of two matrices \mathbf{A} and \mathbf{B} is defined only if
 - the number of columns of \mathbf{A} is equal to the number of rows of \mathbf{B} .
- If \mathbf{A} is $(m \times p)$ and \mathbf{B} is $(p \times n)$, the product is an $(m \times n)$ matrix

\mathbf{C}

$$\mathbf{C}_{m \times n} = \mathbf{A}_{m \times p} \mathbf{B}_{p \times n}$$

Matrix Multiplication

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & & a_{2p} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + \cdots + a_{1p}b_{p1} & a_{11}b_{12} + \cdots + a_{1p}b_{p2} & \cdots & a_{11}b_{1n} + \cdots + a_{1p}b_{pn} \\ a_{21}b_{11} + \cdots + a_{2p}b_{p1} & a_{21}b_{12} + \cdots + a_{2p}b_{p2} & & a_{21}b_{1n} + \cdots + a_{2p}b_{pn} \\ \vdots & & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mp}b_{p1} & a_{m1}b_{12} + \cdots + a_{mp}b_{p2} & \cdots & a_{m1}b_{1n} + \cdots + a_{mp}b_{pn} \end{pmatrix}$$

Example - Matrix Multiplication

$$C = A \cdot B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 2 & 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 1 \\ 2 \cdot 2 + 1 \cdot 1 + 4 \cdot 2 & 2 \cdot 1 + 1 \cdot 2 + 4 \cdot 1 \\ 1 \cdot 2 + 4 \cdot 1 + 3 \cdot 2 & 1 \cdot 1 + 4 \cdot 2 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 13 & 8 \\ 12 & 12 \end{bmatrix}$$

$$C = A \cdot B = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 12 & 21 \\ 10 & 15 \\ 32 & 51 \end{bmatrix}$$

Diagonal Matrices

- Diagonal Matrix $A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$

- Identity Matrix $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- The identity matrix has the property that if A is a square matrix, then

$$IA = AI = A$$

Matrix Inverse

- If \mathbf{A} is a square matrix and there is a matrix \mathbf{X} with the property that $\mathbf{AX} = \mathbf{I}$

- \mathbf{X} is defined to be the *inverse* of \mathbf{A} and is denoted \mathbf{A}^{-1}

$$\mathbf{AA}^{-1} = \mathbf{I} \qquad \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

- Example 2x2 matrix inverse

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Summary

- Matrices
- Matrix Arithmetic
 - Addition
 - Multiplication
- Diagonal Matrices
 - Identity matrix
- Matrix Inverse