Matrices

CE 311 K - Introduction to Computer Methods

Daene C. McKinney

Introduction

- Matrices
- Matrix Arithmetic
  - Addition
  - Multiplication
- Diagonal Matrices
  - Identity matrix
- Matrix Inverse

Matrix

- Matrix - a rectangular array of numbers arranged into *m* rows and *n* columns:

$$\text{Rows, } i = 1, \ldots, m \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \right.$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{Columns, } j = 1, \ldots, n}$$

Matrices

- Variety of engineering problems lead to the need to solve systems of linear equations

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

matrix **column vectors**

Row and Column Matrices (vectors)

- Row matrix (or row vector) is a matrix with one row

$$\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & r_3 & \cdots & r_n \end{pmatrix}$$

- Column vector is a matrix with one column

$$\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

Square Matrix

- When the row and column dimensions of a matrix are equal (m = n) then the matrix is called square

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Matrix Transpose

- The *transpose* of the (m x n) matrix $\boldsymbol{A}$ is the (n x m) matrix formed by interchanging the rows and columns such that row i becomes column i of the transposed matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad \boldsymbol{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & & \ddots & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Matrix Equality

- Two (m x n) matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are *equal* if and only if each of their elements are equal. That is

$\boldsymbol{A} = \boldsymbol{B}$

if and only if

$a_{ij} = b_{ij}$ for $i = 1,\ldots,m$; $j = 1,\ldots,n$

Vector Addition

- The sum of two (m x 1) column vectors ***a*** and ***b*** is

$$\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_m + b_m \end{pmatrix}$$

Matrix Addition

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & & a_{2n} + b_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

Matrix Multiplication

- The *product* of two matrices ***A*** and ***B*** is defined only if
  - the number of columns of ***A*** is equal to the number of rows of ***B***.
- If ***A*** is (m x p) and ***B*** is (p x n), the product is an (m x n) matrix ***C***

$$\boldsymbol{C}_{mxn} = \boldsymbol{A}_{mxp}\,\boldsymbol{B}_{pxn}$$

Matrix Multiplication

$$\boldsymbol{C} = \boldsymbol{AB} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & & a_{1p} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{1n} \\ \vdots & & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + \cdots + a_{1p}b_{p1} & a_{11}b_{12} + \cdots + a_{1p}b_{p2} & \cdots & a_{11}b_{1n} + \cdots + a_{1p}b_{pn} \\ a_{21}b_{11} + \cdots + a_{2p}b_{p1} & a_{21}b_{12} + \cdots + a_{2p}b_{p2} & & a_{21}b_{1n} + \cdots + a_{2p}b_{pn} \\ \vdots & & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mp}b_{p1} & a_{m1}b_{12} + \cdots + a_{mp}b_{p2} & \dots & a_{m1}b_{1n} + \cdots + a_{mp}b_{pn} \end{pmatrix}$$

Example - Matrix Multiplication

$$C = A \cdot B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 2 + 2\cdot 1 + 3\cdot 2 & 1\cdot 1 + 2\cdot 2 + 3\cdot 1 \\ 2\cdot 2 + 1\cdot 1 + 4\cdot 2 & 2\cdot 1 + 1\cdot 2 + 4\cdot 1 \\ 1\cdot 2 + 4\cdot 1 + 3\cdot 2 & 1\cdot 1 + 4\cdot 2 + 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 13 & 8 \\ 12 & 12 \end{bmatrix}$$

$$C = A \cdot B = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 12 & 21 \\ 10 & 15 \\ 32 & 51 \end{bmatrix}$$

Diagonal Matrices

- Diagonal Matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$

- Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The identity matrix has the property that if A is a square matrix, then

$$IA = AI = A$$

Matrix Inverse

- If ***A*** is a square matrix and there is a matrix ***X*** with the property that $AX = I$
- ***X*** is defined to be the *inverse* of ***A*** and is denoted A^{-1}

$$AA^{-1} = I \qquad A^{-1}A = I$$

- Example 2x2 matrix inverse

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Summary

- Matrices
- Matrix Arithmetic
  - Addition
  - Multiplication
- Diagonal Matrices
  - Identity matrix
- Matrix Inverse