Computation of Slope

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There are various ways in which slope can be computed depending on the nature of the information available and the method to be used.

A Line in a 3D space

The simplest way to think about slope is to consider two points in a three dimensional space and a line that connects them. Suppose there are two such points, A and B, and they have coordinates $(x_a, y_a, z_a,)$ and $(x_b, y_b, z_b,)$. If we connect them by a line drawn from A to B then the components of the vector AB are $(\Delta x, \Delta y, \Delta z)$, where $\Delta x = (x_b - x_a)$, $\Delta y = (y_b - y_a)$, $\Delta z = (z_b - z_a)$. The vector AB can also be thought of as having two components, a *rise*, which represents the vertical difference, Δz, between the two points, and a *run*, $\sqrt{\Delta x^2 + \Delta y^2}$, which represents the horizontal distance between them. With these definitions, the slope of the line AB is given by

$$Slope = \frac{Rise}{Run} \qquad (1)$$

In addition, the *azimuth* of the line AB, or its orientation angle measured clockwise from North, is given by

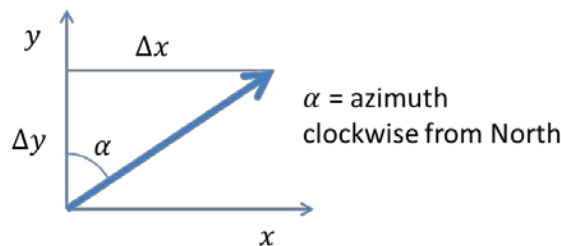$$Azimuth = \frac{180}{\pi} tan^{-1} \frac{\Delta x}{\Delta y} \qquad (2)$$

Figure 1. Azimuth angle measured clockwise from North

Note that due to angles being measured from the y-axis (North), the ratio in (2) is $\Delta x / \Delta y$, not the more common $\Delta y / \Delta x$ used for angles from the x-axis.

Example

The location of two survey benchmarks on 24th St on the University of Texas campus is shown in Figure 2, and Figure 3 shows a photograph of Benchmark B which is located on a bridge over Waller Creek.   These benchmarks serve as reference points for the location of campus facilities. The coordinates of these two benchmarks with all dimensions in feet are given by

Benchmark	Easting	Northing	Elevation
A	819470.95	235256.11	555.42
B	820462.32	235187.15	524.06

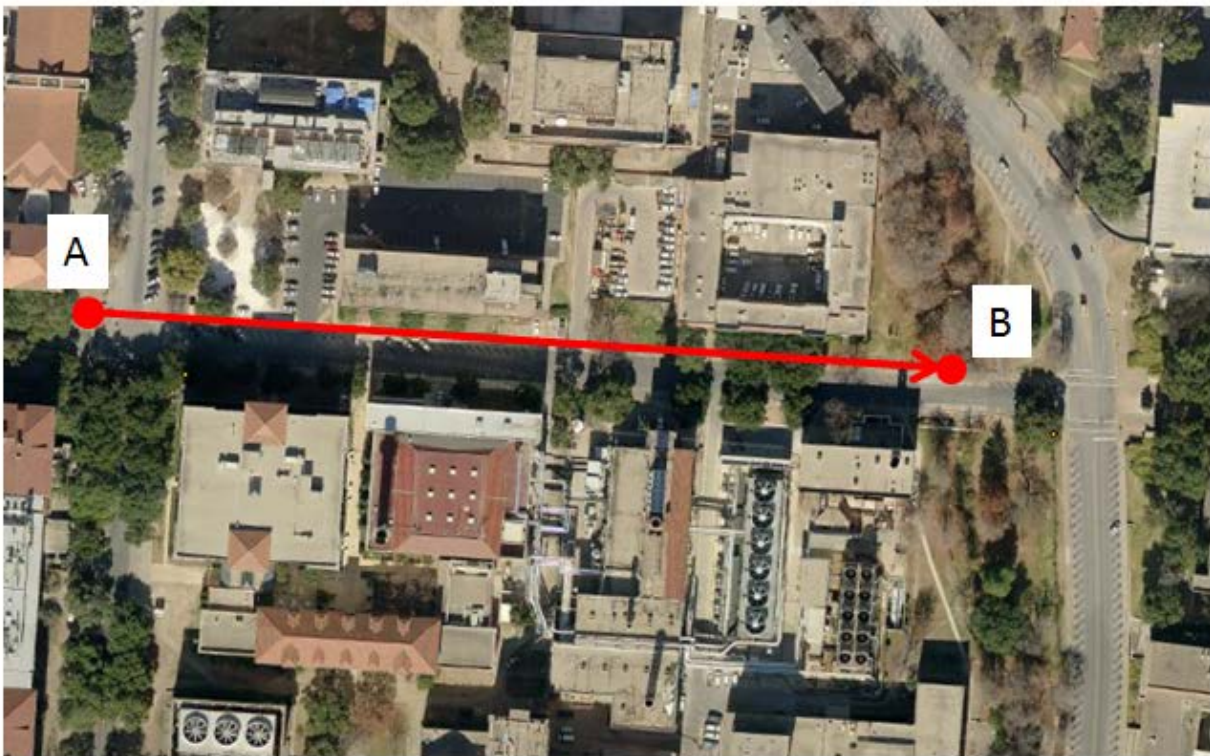Determine the length, slope and azimuth of the line AB.

Figure 2. Survey benchmarks on the University of Texas campus

Figure 3. Location of Survey Benchmark 2

Solution

In the coordinate values, the Easting is x, Northing is y, and Elevation is z, so the x-component of the vector AB is given by Δx = 820462.32 - 819470.95 = 991.37, and similar computations for Δy and Δz produce the following values in feet:

	Δx	Δy	Δz
AB	991.37	-68.96	-31.36

Notice that while Δx is positive meaning that B is East of A, Δy is negative meaning that B is slightly South of A, and Δz is also negative because B is below A, and in fact water flows down 24th St from A to B as it drains towards Waller Creek on our campus.

The Length of the line AB is given by

$$Length = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

$$= \sqrt{991.37^2 + (-68.96)^2 + (-31.36)^2}$$

$$= \mathbf{994.26\ ft}$$

For the line AB, the rise = -31.36 ft, and the run = $\sqrt{991.37^2 + (-68.96)^2} = 993.77$ ft, and substituting these quantities into Eq. (1),

$$\text{Slope} = \frac{-31.36}{993.77}$$

$$\text{or,} \quad \text{Slope} = -0.0316$$

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This means that the average downhill slope of 24th St as water drains along it towards Waller Creek is a little more than 3%. Another way of expressing the slope is as an angle in degrees, called the zenith, which may be computed as

$$\text{Zenith} \;=\; \frac{180}{\pi}\tan^{-1}(-0.0316)$$

$$\text{or,} \quad \text{Zenith} \;=\; -1.807°$$

Similarly, by substituting into Eq. (2), the azimuth of line AB is given by

$$\text{Azimuth} = \frac{180}{\pi}\tan^{-1}\frac{\Delta x}{\Delta y}$$

However, the computation of the azimuth is more complicated mathematically than for the zenith because the inverse tangent function is multi-valued. This does not matter for the zenith angle which is limited to the range -90⁰ to 90⁰. However for the azimuth angle which has the range 0⁰ to 360⁰, this multi-valuedness is an issue because the ratio $\Delta x/\Delta y$ is the same as the ratio $(-\Delta x)/(-\Delta y)$, but vectors $(\Delta x,\ \Delta y)$ and $(-\Delta x, -\Delta y)$ are in opposite directions. The atan2 function has been developed to resolve this ambiguity. In Excel the atan2 function is defined as follows: "atan2(x_num,y_num) returns the arctangent, or inverse tangent, of the specified x- and y-coordinates. The arctangent is the angle from the x-axis to a line containing the origin (0, 0) and a point with coordinates (x_num, y_num). The angle is given in radians between -pi and pi, excluding –pi". Note that computer programming languages commonly define atan2 differently, with the arguments reversed from Excel, so in using this function you need to check on the convention used. Wikipedia provides a thorough discussion (http://en.wikipedia.org/wiki/Atan2).

In this instance, we have

$$\text{Azimuth} \;=\; \frac{180}{\pi}\text{atan2}\,(\Delta y, \Delta x) \;=\; \frac{180}{\pi}\text{atan2}\,(-68.96, 991.37) = 93.98°$$
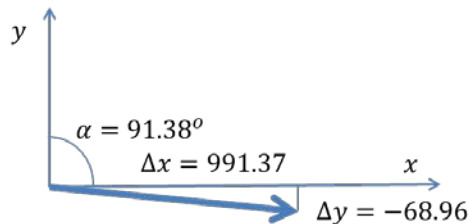

Figure 4. Azimuth of line AB

Note again that in the evaluation of atan2, the order of arguments is $\Delta y, \Delta x$ because we are measuring the angle from the y-axis (North). The atan2 function in Excel returns a value

between $-\pi$ and π, so here with 180/π as a pre-multiplier the range is -180° to 180°. In case a negative value is obtained add 360° to obtain the corresponding angle clockwise from North.

So, in summary, there several ways that we can think about the definition of the line AB:

A vector that originates at point A, and has components in the (x,y,z) Cartesian coordinate space of (991.37, -68.96, -31.36).

A vector of length 994.26 ft that originates at A, which is rotated 93.98° from North in the horizontal plane, and then – 1.81° in the vertical plane to reach the point B. The values (994.26, 93.98, -1.81) are its coordinates in a Spherical coordinate space.

A vector drawn in the azimuth direction of 93.98° that has a horizontal component or run of 991.37 ft and a vertical component or rise of -31.36 ft.

In all these instances, it takes three pieces of information to define the line that originates at point A and ends at point B in a 3D space, in the first instance three length values, in the second instance one length value and two angles, and in the third instance two length values and one angle. These are all valid ways of thinking about how vectors are defined in a 3D space, but the third one is the most useful for defining the slope of the line.

A Function in a 2D Space

Suppose we have a 2D space (x,y) where x and y are coordinates in the horizontal plane, and we have some arbitrary function F defined over this plane. The value of F could represent any scalar quantity, such as annual precipitation or air temperature. If we wish to define the gradient of this function at a particular location (x,y), we would use the nomenclature:

$$\nabla F(x, y) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right)$$

where $\partial F / \partial x$ represents the partial derivative of the function in the x-direction, $\partial F / \partial y$ represents the partial derivative of the function in the y-direction, each evaluated at the point (x,y). In this case, the magnitude of the gradient is given by:

$$|\nabla F(x, y)| = \sqrt{\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2}$$

And the direction of the gradient is given by

$$\text{Direction}[\nabla F(x, y)] = \text{atan2} \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right)$$

Land Surface Slope

Land surface elevation is a function defined on a 2D coordinate space, that is, at any location (x,y) on the land portion of the earth's surface, there is a corresponding elevation value, let's call it Z, which represents the vertical height of the land surface above the geoid surface at that location. Z is capitalized to distinguish the land surface elevation from any quantity defined by a z-coordinate. Now we can define the gradient, or slope of the land surface by a 2D vector:

$$\text{Land Surface Slope} = \nabla Z(x, y) = \left(\frac{\partial Z}{\partial x}, \frac{\partial Z}{\partial y} \right)$$

The magnitude of the land surface slope is given by

$$|\text{Slope}| = \sqrt{\left(\frac{\partial Z}{\partial x} \right)^2 + \left(\frac{\partial Z}{\partial y} \right)^2}$$

To make this computation meaningful, Z needs to be defined in the same length units as x and y, and the result is a dimensionless quantity that represents the slope with positive being upwards. When multiplied by 100, this becomes the percent rise value arising from the slope function in the ArcGIS software. A computation corresponding to that for the zenith angle described earlier gives the slope angle in degrees in the ArcGIS Slope function.

In physical geography, the orientation of the land surface is defined by its aspect, which is analogous to the azimuth used in land surveying. For example, we might describe mountainside as having a Westerly aspect if the ridge at its crest lies in the North-South direction, and the land sloping down from that on one side, faces towards the West (and correspondingly an Easterly aspect on the other side of the ridge). By analogy with the azimuth, the ArcGIS aspect function evaluates aspect as:

$$\text{Aspect} = \text{atan2} \left(\frac{\partial Z}{\partial y}, \frac{\partial Z}{\partial x} \right)$$

Aspect identifies the downslope direction so the partial derivatives in this equation need to be evaluated as positive downslope. ArcGIS uses finite difference approximations for these partial derivatives

a	b	c
d	e	f
g	h	i

Figure 5. Mesh used to approximate partial derivatives

$$\frac{\partial z}{\partial x} \approx \frac{(a + 2d + g) - (c + 2f + i)}{8 * x\_mesh\_spacing}$$

$$\frac{\partial z}{\partial y} \approx \frac{(g + 2h + i) - (a + 2b + c)}{8 * y\_mesh\_spacing}$$

The basis for this is illustrated in Figure 6 and amounts to an average of central finite differences over each of the three rows of cells with the middle row counting twice as it appears in averages on each side.
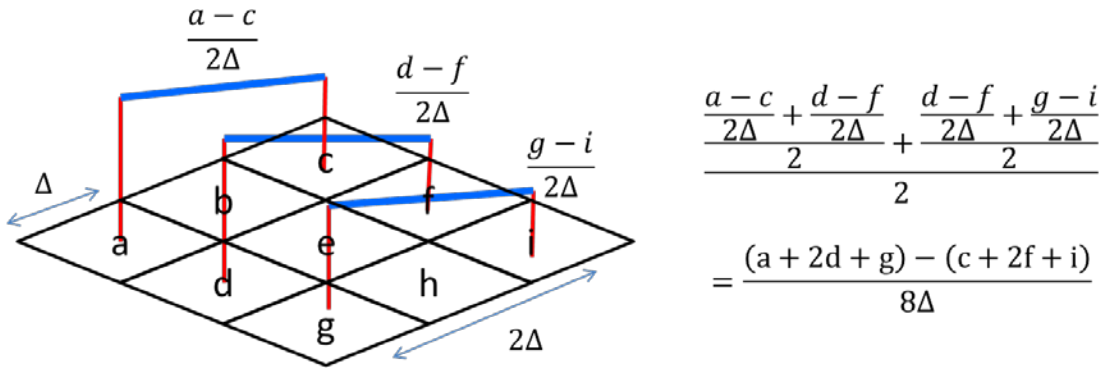

$$\frac{\frac{a-c}{2\Delta} + \frac{d-f}{2\Delta}}{2} + \frac{\frac{d-f}{2\Delta} + \frac{g-i}{2\Delta}}{2}}{2}$$

$$= \frac{(a + 2d + g) - (c + 2f + i)}{8\Delta}$$

Figure 6. Basis for slope finite difference approximation

Hydrologic Slope

To define slope in hydrology, we are interested in the pathway of water flowing downhill.   This means that slope is considered a positive quantity if pointing downhill.   The D8 flow direction is a special case, because although it is being defined on a continuously gridded elevation surface, only 8 discrete flow directions are possible, and so effectively, the D8 Slope is computed as

$$\text{D8 Slope} = \frac{\text{Drop}}{\text{Run}}$$

Where drop refers to the difference in elevation between the "from cell" and the "to cell", with downwards considered positive, and the run is equal to the cell size for flow along the coordinate directions, and equal to $\sqrt{2}$ x cell size for flow in the diagonal directions. So, in effect, the D8 flow direction is a slope computation on a set of eight 3D lines like the lines connecting the survey benchmarks in the example described above, where the one giving the maximum downward slope is selected as the direction of steepest descent and its corresponding value (1, 2, 4,…., 64, 128) is encoded in the flow direction grid.

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Slope of a River

Suppose we have a river defined on a map as a sequence of stream segments, and we wish to calculate the slope of the river.   The picture below shows the Logan River and basin above Logan, Utah, and a digital elevation model of the area.   The length of the river is found by adding up the length of each of its segments.
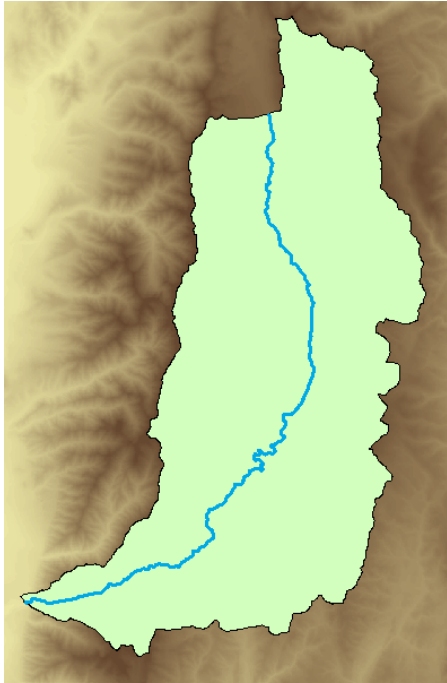
In this case the river segments come from the NHDPlus dataset, and these have a LengthKm attribute on them, so the length of the river itself can be found by summing this attribute to yield a result of Length = 53.917 km.
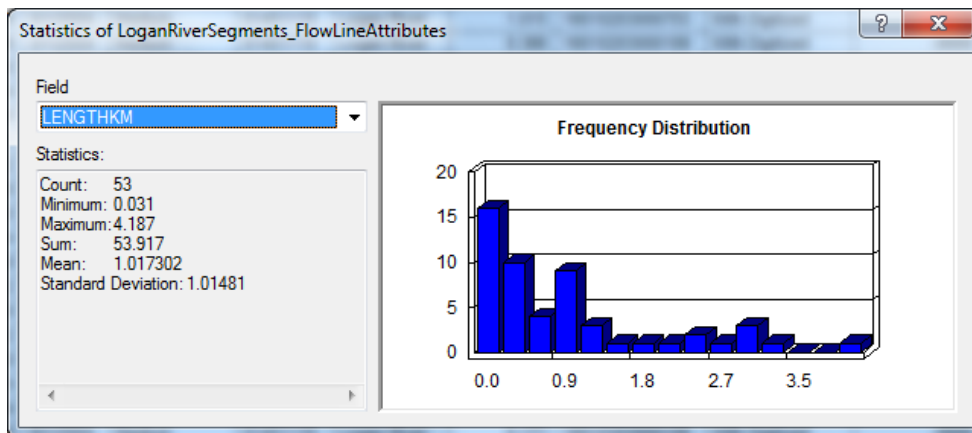
A simple way to compute the slope of the river is to dissolve all the stream segments into a single feature for the river, and use the function ZonalStats as Table to find the minimum and maximum elevations that lie under that feature.   For the Logan River, this results in the table:

Table

LoganRiverProperties

OBJECTID *	LINK	COUNT	AREA	MIN	MAX	RANGE	MEAN	STD	SUM
1	1	2505	1811362.5	1425.8148	2466.4871	1040.6722	1842.823	287.74677	4616271.5

|◄ ◄   1 ► ►| (0 out of 1 Selected)

LoganRiverProperties

Thus, the maximum elevation is 2466.49m and minimum elevation is 1425.81m, so the elevation difference is 1040.68m, and the slope of the river is 1040.68/53917 m/m = 0.0193 or 1.93%.

This is an elaborate version of the point to point calculation of slope described earlier where now the length of the line is along a river rather than simply being a geometrically straight line.

Now, it could occur that the slope of the river changes as you go down the river (steeper in the headwaters, and flatter downstream). One way to allow for this is to compute a weighted average of slope, S_i, of each stream segment, where the weights are the lengths of the stream segments, L_i. The FlowLineAttributes flow table of NHDPlus has a Slope attribute, so the overall river slope S_R, is computed as:

$$S_R = \frac{\sum_{i=1}^{n} L_i S_i}{\sum_{i=1}^{n} L_i}$$

In the case of the Logan river, this computation yields S_R = 1.0589/53.917 = 0.01964, or 1.96%, almost the same as the previous result. Here is a picture of the slopes of some individual stream segments:

0.01801

0.01384

0.00895

0.00936

0.00776

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