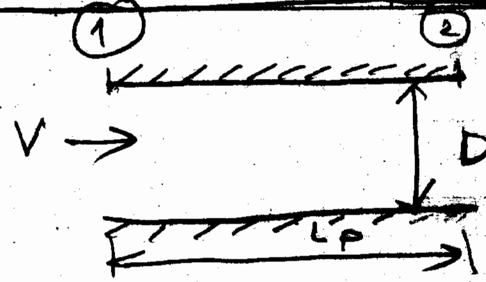
Example of Dimensional Analysis on pipe flows



$$h_L = f \frac{L}{D} \frac{V^2}{2g} \quad (V \equiv \bar{V})$$

↳ $f(Re, k_s/D)$

How did we come up with this?

$\Delta p = p_1 - p_2$

$h_L = \frac{p_1 - p_2}{\gamma}$ ↑ (for horizontal pipes)

$$\frac{\Delta p}{L_p} = f(D, V, \mu, \rho)$$

If we wanted to do experiments for a large range of parameters: D, V, μ, ρ we would need say: $10 \times 10 \times 10 \times 10 = 10^4$ experiments!
(if we also included 10 k_s's that would be 10^5 exp.!)

It is important to <u>reduce</u> <u>independent</u> <u>parameters</u> ⇒ <u>dimensional</u> <u>analysis</u>
(Buckingham π- theorem)

First we express all variables in terms of $[L], [T], [M]$ basic dimensions (or more, e.g. T = temperature, if needed)
length time mass

$$\frac{\Delta p}{L_p} = \frac{(N/m^2)}{m} = \frac{N}{m^3} = \frac{kg\ \cancel{m}/s^2}{m^{\cancel{3}}\,2} = \frac{kg}{m^2 s^2} = \frac{[M]}{[L]^2 [T]^2}$$

$D \rightarrow [L]$

$V \rightarrow [L]/[T]$

$$\mu \rightarrow Pa.s. = \frac{N}{m^2} \cdot s = \frac{kg\ \frac{\cancel{m}}{s^2}}{m^{\cancel{2}}} \cancel{s} = \frac{kg}{m \cdot s} = \frac{[M]}{[L][T]}$$

$$\rho \rightarrow \frac{kg}{m^3} = \frac{[M]}{[L]^3}$$

$n = \#$ of variables $= 5$

$m =$ number of basic (independent parameters) $= 3$

$n - m = 5 - 3 = 2$ π-groups $= 2$ groups of non-dimensional parameters, i.e.

$$\pi_1 = \varphi(\pi_2)$$

2 methods to determine parameters:

1) step-by-step method:

$\Delta P / L_P$	$\frac{[M]}{[L]^2[T]^2}$	$\frac{\Delta P}{L_P} D^2$	$\frac{[M]}{[T]^2}$	$\frac{\Delta P}{L_P} D^2 \frac{D^2}{V^2}$	$[M]$	$\frac{\Delta P}{L_P} \frac{D^4}{V^2} \frac{1}{\rho D^3}$
D	$[L]$	—	—	—	—	—
V	$[L]/[T]$	V/D	$1/[T]$	—	—	—
μ	$[M]/[L][T]$	μD	$[M]/[T]$	$\mu D \frac{D}{V}$	$[M]$	$\mu D^2/V \times 1/\rho D^3$
ρ	$[M]/[L]^3$	ρD^3	$[M]$	ρD^3	$[M]$	—

$$\Pi_1 = \frac{\Delta P}{L_P} \frac{D}{\rho V^2} \qquad \Pi_2 = \frac{\mu D^2}{V} \frac{1}{\rho D^3} = \frac{\mu}{V \rho D} = \frac{\nu}{VD} = \frac{1}{Re}$$

$$\frac{\Delta P}{L_P} \frac{D}{\rho V^2} = \varphi(Re)$$

Remember:
$h_L = \frac{\Delta P}{\gamma}$
(for horizontal pipe)

$$\frac{\Delta P}{\left(\frac{L_P}{D}\right)} \frac{1}{\frac{\gamma}{g} V^2} = \varphi(Re)$$

$$\frac{(\Delta P)/\gamma}{(L_P/D)} \frac{1}{\frac{V^2}{2g} 2} = \varphi(Re) \Rightarrow h_L = \underbrace{2\varphi(Re)}_{f} \frac{L_P}{D} \frac{V^2}{2g}$$

f = friction factor

2) Exponent method:

$$\frac{\Delta p}{L_p} = D^a V^b \mu^c \rho^d$$

meant dimensionally (i.e. dimensions of LHS same as those of RHS)

$$\frac{[M]}{[L]^2[T]^2} = [L]^a \frac{[L]^b}{[T]^b} \cdot \frac{[M]^c}{[L]^c \cdot [T]^c} \cdot \frac{[M]^d}{[L]^{3d}}$$

$$= [L]^{a+b-c-3d} [T]^{-b-c} [M]^{c+d}$$

$$\begin{bmatrix} c+d=1 \\ a+b-c-3d=-2 \\ -b-c=-2 \end{bmatrix}$$

- 4 unknowns
- 3 equations

$\Rightarrow$ make one of the unknowns an independent parameter e.g. $\underline{c}$

$\underline{d = 1-c}$

$\underline{b = 2-c}$

(after substituting d, b)

$$\Rightarrow a + (2-c) - c - 3(1-c) = -2 \Rightarrow$$

$$\Rightarrow a + 2 - c - c - 3 + 3c = -2 \Rightarrow$$

$$\Rightarrow a + c - 1 = -2 \Rightarrow \underline{a = -c-1}$$

Then: $$\frac{\Delta p}{L_p} = D^{-c-1} V^{2-c} \mu^c \rho^{1-c} =$$

$$= D^{-1} V^2 \rho \; D^{-c} V^{-c} \mu^c \rho^{-c} =$$

$$= \rho \frac{V^2}{D} \cdot \left(\frac{\mu}{D V \rho}\right)^c = \rho \frac{V^2}{D} \left(\frac{1}{Re}\right)^c$$

$$\Rightarrow \underbrace{\frac{\Delta p}{L_p} \frac{D}{\rho V^2}}_{\Pi_1} = \varphi\left(\underbrace{Re}_{\Pi_2}\right)$$

same final answer as from the step-by-step method!