Derivation of perturbation curvilinear methods

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Abstract

This technical report presents detailed derivations of a perturbation curvilinear approach that can be applied to numerical modeling of rivers, estuaries and reservoirs wherein the channel width is small compared to the radius of curvature of bends. The report complements the manuscript by Hodges and Imberger: "A perturbation curvilinear form of the Navier-Stokes equations," Centre for Water Research ED1120BH (2000).

Contents

Chapter 1

Introduction

1.1 2D v. 3D modelling

Hydrodynamic models of rivers and estuaries can often produce useful analyses of currents and transport mechanisms by using two-dimensional vertically-averaged or laterally-averaged methodology. However, it can be argued that coupled hydrodynamic/water-quality modelling of estuaries and rivers should be fundamentally approached as a three-dimensional system that is not particularly amenable to reduction in dimension. In particular, lateral averaging (typically used in riverine systems) produces a model which has a uniform depth at each cross section, eliminating the effects of the shallows where reduced water velocities, nutrient introduction from land margins and the high light levels at the benthic boundary result in ideal conditions for algal production. Using vertical averaging may preserve some of the shallow effects that are lost in laterally-averaged systems, but produces a distorted picture of the transport when stratification effects (either temperature or salinity) allow the development of significant baroclinic modes of motion.

1.2 Time scales

Arriving at the idea that three-dimensional models of hydrodynamics are desirable for water-quality modelling, we are presented with the problem of disparities in time scales. Water quality responds to changes in environmental forcing on relatively long time scales (days, weeks or months) while the hydrodynamics responds rapidly (minutes or hours) to heating, cooling and changes in flow rate.

The maximum time step in a hydrodynamic model is fundamentally limited by the flow rate and the space scale on which the regime is gridded. Even without consideration of the type of numerical model applied, it holds true that finer grid scales demand finer time steps. It follows that the manner in which we produce a computational grid will influence the allowable time step in a model and thus will determine the temporal length of a model run that can be achieved with a fixed amount of computational power.

1.3 Space scales

Estuaries are rivers are typically characterized by a significant disparity in cross-channel and along-channel length scales. That is, the cross-channel dimension may range from tens or hundreds of metres to several kilometres, while the along-channel dimension may range from tens to hundreds of kilometres. A grid scale appropriate to resolving the along-channel physics and water quality may be on the order of 500 metres to 2 or perhaps 3 kilometres (depending on the system), which is clearly inappropriate for a cross-channel grid. Typically we find that the appropriate grid-spacing in the cross-channel direction is in the range of 10 to 50 metres. To this point, previous 3D works have addressed the problem of the disparity in spatial scales in one of three ways: (1) boundary-conforming structured curvilinear grids, (2) uniform Cartesian grids applying the cross-channel grid scale over the entire domain, (3) finite element algorithms applied on triangular meshes. All these approaches are capable of producing 3D models, but each has peculiar drawbacks.

1.3.1 Boundary-conforming curvilinear grids

The use of boundary-conforming structured curvilinear grids allows computationally efficient finite-difference models to be used but has three major drawbacks: (1) development of a boundary-fitted curvilinear grid for a topologically complex estuary is not a trivial task; (2) the numerical discretization for solution of the Navier-Stokes equations on curvilinear coordinates are significantly more complex than that required for a simple Cartesian system; and (3) the time step of the curvilinear solution is generally set by the smallest of the curvilinear grid cells: the compression of the grid in a narrow channel may limit the time step over the entire domain. The last drawback can certainly be addressed by the use of effective grid nesting for small features, and the first drawback can be addressed by the use of suitable domain-decomposition techniques. However, both these add to the computational complexity of the modelling task.

1.3.2 Uniform Cartesian grids

The use of uniform Cartesian grids (using the cross-channel grid scale) allows application of computationally efficient models to be used with complicated topography in what might be termed a "naive" manner. That is, we simply discretize the domain with some suitable grid whose size is determined by the smallest feature we would like to resolve. As this is typically based on obtaining 5 to 10 grid cells in the cross-channel direction, this approach requires for an inordinately large amount of grid cells to discretize an estuary. For example, the use of a 20×20 metre horizontal grid to discretize the upper 24 kilometres of the Swan River estuary with 10 grid cells in the vertical direction requires a total of 2×10^5 grid cells. In addition, the use of a uniformly fine mesh throughout the domain requires a small time step and reduces the practicality of using uniform Cartesian grids for seasonal computation.

1.3.3 Finite-element models

Finite-element models have some significant advantages in the ability to easily grid a complex topographical space with triangles. Furthermore, the existence of well-tested, commercial finite-element flow solvers that are inherently stable for large time steps can be attractive to the casual user. The drawbacks of the finite-element approach are the computational complexity of the algorithms and the relatively high demands of computer memory and CPU time required for unsteady flow computations. It has yet to be demonstrated that a finite-element method can be competitive with a finite-difference method in CPU time per real-time interval for models with similar grid resolutions. Furthermore, one must be careful not to confuse stability at large time steps with accuracy. While it is perfectly possible to design a numerical method that is stable for a CFL ¿ 10, the accuracy of any such algorithm is very much in question. If the time scale of the model is significantly larger than the fundamental time scales of the unsteady physics, then there cannot be an accurate solution.

1.3.4 A proposal for a "straightened" Cartesian grid

In this report, we develop an approach that has the advantages of the uniform Cartesian grid (simplicity and efficiency of algorithms) while allowing different grid scales in cross-river and along-river directions in a manner similar to boundary-fitted curvilinear coordinate systems. To the "zeroth" order, this approach is a simple straightening of the river or estuary so that a rectangular Cartesian grid can be applied. It will be demonstrated that this is identical to a curvilinear transformation that neglects terms that have the leading order of $\delta\, r_c^{-1}$, where r_c is the radius of curvature at the center of the river or estuary and δ is the half-channel width. As $\delta\, r_c^{-1}$ is a small number throughout most estuaries and rivers, we can include terms of this order (and smaller) as source/sink terms in the Navier-Stokes equations and thus make simple modifications to a Cartesian-grid model to account for the curvilinear effects.

Chapter 2

A grid-stretched curvilinear form of the Navier-Stokes equations

Our objective is to define a curvilinear form of the Navier-Stokes equations that can be seen as the Cartesian form of the equations plus perturbation terms. The curvilinear derivations in this paper rely on the tensor concepts found in Aris (1962) and apply the Einstein summation convention to repeated subscripts placed in contravariant/covariant pairs.

2.1 Definitions of curvilinear terms

Consider the transformation between Cartesian space (x^i or x, y, z) and curvilinear space (ξ^q or ξ, η, ζ) where the covariant transformation metrics are defined as:

$$R^i_q \equiv \frac{\partial x^i}{\partial \xi^q} \tag{2.1}$$

and the covariant metric tensor (Aris, 1962, eq 7.23.4)

$$G_{qr} \equiv \sum_{j=1}^{3} R^j_q R^j_r \tag{2.2}$$

and the Jacobian of the transformation from Cartesian to curvilinear space is

$$J \equiv \det \left| \frac{\partial x^i}{\partial \xi^j} \right| \tag{2.3}$$

In some texts, this would be defined as the inverse Jacobian. For the purposes of this paper, we shall use the above convention as found in Aris (1962, eq 7.24.4). The contravariant transformation metric tensor can be defined in terms of the covariant tensor (Aris, 1962, eq 7.24.8)

$$G^{ij} = \frac{1}{2\,J^2}\,\epsilon^{imn}\epsilon^{jpq}\,G_{mp}G_{nq} \tag{2.4}$$

Where ϵ^{imn} is the tensor permutation symbol. The unsteady incompressible Navier-Stokes equations can be written in the tensor form found in Aris (1962, eq 8.22.2)

$$\rho \left\{ \frac{\partial U^q}{\partial t} + U^j U^q_{,j} \right\} = -G^{qj} p_{,j} + \mu\, G^{jk}\, U^q_{,jk} \tag{2.5}$$

Where the tensor derivatives (i.e. $U^q_{,j}$) indicate covariant differentiation (requiring Christoffel symbols for evaluation). Note that the scalars ρ, μ and p are defined as properties of physical space and are unaffected by the transformation. If we let $\zeta = f(z)$, then the unsteady incompressible Navier-Stokes equations with the hydrostatic and Bousinesq approximations can be written as

$$\frac{\partial U^\alpha}{\partial t} + U^j U^\alpha_{,j} \;=\; -\,g\,G^{\alpha\beta}\,H_{,\alpha} - \frac{g\,G^{\alpha\beta}}{\rho_0}\left\{\int_{z'}^{H}\rho'\,dz\right\}_{,\beta} + \nu\,G^{jk}\,U^\alpha_{,jk} \tag{2.6}$$

where Latin sub- and super-scripts are evaluated over 3-space (i.e. $j,k = 1,2,3$) while Greek sub- and super-scripts are evaluated over 2-space (i.e. $\alpha,\beta = 1,2$). Slightly more clearly, this can be written as

$$\frac{\partial U^\alpha}{\partial t} + U^j U^\alpha_{,j} \;=\; -\,g\,G^{\alpha\beta}\frac{\partial H}{\partial \xi^\beta} - \frac{g\,G^{\alpha\beta}}{\rho_0}\frac{\partial}{\partial \xi^\beta}\int_{z'}^{H}\rho'\,dz + \nu\,G^{jk}\,U^\alpha_{,jk} \tag{2.7}$$

2.2 Covariant metrics

Consider the definition of grid terms provided in figure (2.1). The angles θ and ϕ are the angles that the curvilinear ξ and H axes form with the x axis. The axes h_1 and h_2 are a discrete linear version of the curvilinear axes that are measured in physical space dimensions such that $\Delta h^2 = \Delta x^2 + \Delta y^2$. From some simple trigonometry we can write:

$$\begin{aligned}\frac{\Delta x_1}{\Delta h_1}\frac{\Delta x_2}{\Delta h_2} + \frac{\Delta y_1}{\Delta h_1}\frac{\Delta y_2}{\Delta h_2} &= \cos\theta\,\cos\phi + \sin\theta\,\sin\phi \\ &= \cos(\theta - \phi)\end{aligned} \tag{2.8}$$

Let the local grid skewness is represented by the angle ψ, where

$$\psi \;\equiv\; (\phi - \theta) - \frac{\pi}{2} \tag{2.9}$$

From trigonometry again

$$\cos\left(-\psi - \frac{\pi}{2}\right) \;=\; \sin(-\psi) \;=\; -\sin\psi \tag{2.10}$$

so we end up with

$$\frac{\partial x}{\partial h_1}\frac{\partial x}{\partial h_2} + \frac{\partial y}{\partial h_1}\frac{\partial y}{\partial h_2} \;=\; -\sin\psi \tag{2.11}$$

The chain rule for transformations between ξ^q and h_j can be written as:

$$\frac{\partial}{\partial \xi^q} \;=\; \frac{\partial h_j}{\partial \xi^q}\frac{\partial}{\partial h_j} \tag{2.12}$$

From equation (2.2)

$$\begin{aligned}G_{12} &= \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} \\ &= \frac{\partial h_j}{\partial \xi}\frac{\partial x}{\partial h_j}\frac{\partial h_k}{\partial \eta}\frac{\partial x}{\partial h_k} + \frac{\partial h_j}{\partial \xi}\frac{\partial y}{\partial h_j}\frac{\partial h_k}{\partial \eta}\frac{\partial y}{\partial h_k} \\ &= \frac{\partial h_1}{\partial \xi}\frac{\partial x}{\partial h_1}\frac{\partial h_2}{\partial \eta}\frac{\partial x}{\partial h_2} + \frac{\partial h_1}{\partial \xi}\frac{\partial y}{\partial h_1}\frac{\partial h_2}{\partial \eta}\frac{\partial y}{\partial h_2} \\ &= \frac{\partial h_1}{\partial \xi}\frac{\partial h_2}{\partial \eta}\left\{\frac{\partial x}{\partial h_1}\frac{\partial x}{\partial h_2} + \frac{\partial y}{\partial h_1}\frac{\partial y}{\partial h_2}\right\} \\ &= -\frac{\partial h_1}{\partial \xi}\frac{\partial h_2}{\partial \eta}\sin\psi\end{aligned} \tag{2.13}$$

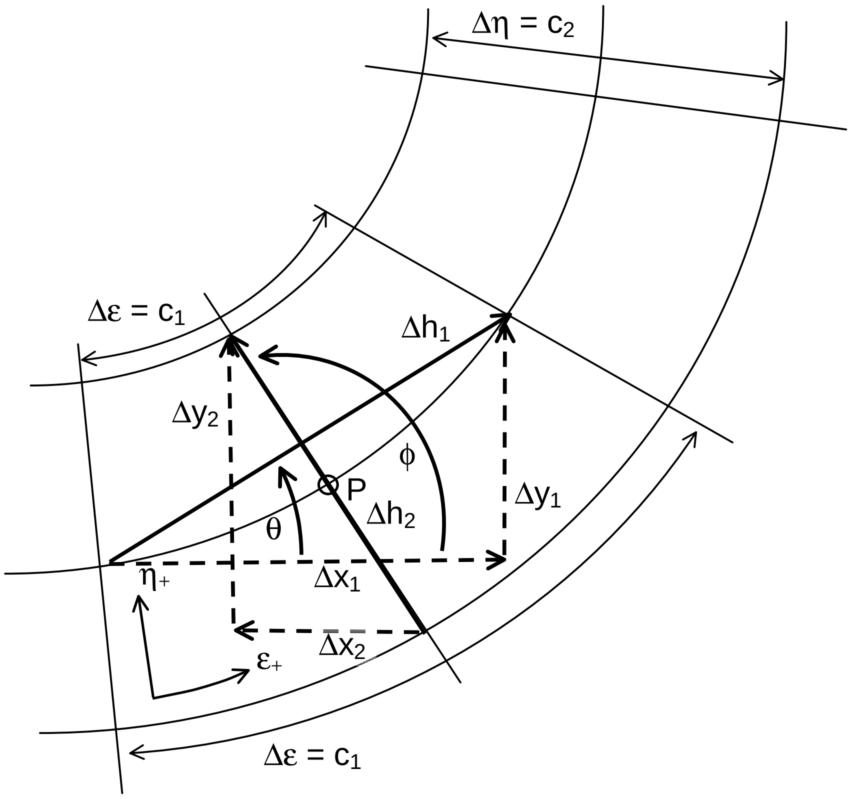

Figure 2.1: Grid definitions: Δh_1, Δh_2, Δx and Δy are physical distances in Cartesian space; $\Delta\xi$ and $\Delta\eta$ are fixed curvilinear coordinate measures.

where we have used the geometrical identity $\partial h_j/\partial\xi^q = 0$ for $j \neq q$ and equation (2.11). Next, from trigonometry we have

$$(dh_\alpha)^2 = (dx)^2 + (dy)^2 \tag{2.14}$$

So that we can write

$$\left(\frac{\partial h_1}{\partial\xi}\right)^2 = \left(\frac{\partial x_1}{\partial\xi}\right)^2 + \left(\frac{\partial y_1}{\partial\xi}\right)^2 = G_{11} \tag{2.15}$$

$$\left(\frac{\partial h_2}{\partial\eta}\right)^2 = \left(\frac{\partial x_2}{\partial\eta}\right)^2 + \left(\frac{\partial y_2}{\partial\eta}\right)^2 = G_{22} \tag{2.16}$$

resulting in equation (2.13) being written as

$$G_{12} = -\sqrt{G_{11}\,G_{22}}\ \sin\psi \tag{2.17}$$

Let us require a transformation that locally preserves physical space dimensions in the horizontal plane with only small amounts of stretching and has no change in the vertical such that:

$$G_{11} = 1 + \gamma_1(x,y) \tag{18.a}$$

$$G_{22} = 1 + \gamma_2(x,y) \tag{18.b}$$

$$G_{33} = 1 \tag{18.c}$$

$$G_{13} = 0 \tag{18.d}$$

$$G_{23} = 0 \tag{18.e}$$

$$G_{12} = -\{1 + \gamma_1 + \gamma_2 + \gamma_1\gamma_2\}^{1/2}\ \sin\psi \tag{18.f}$$

where $\gamma_i = \gamma_i\,(x,\,y)$ and $\psi = \psi\,(x,\,y)$. The Jacobian of the transformation defined by equation (2.3) is

$$\begin{aligned}
J &= \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} \\
&= \frac{\partial h_1}{\partial \xi}\frac{\partial h_2}{\partial \eta}\left\{\frac{\partial x}{\partial h_1}\frac{\partial y}{\partial h_2} - \frac{\partial y}{\partial h_1}\frac{\partial x}{\partial h_2}\right\} \\
&= \sqrt{G_{11}\,G_{22}}\,\{\cos\theta\,\sin\phi - \sin\theta\,\cos\phi\} \\
&= \sqrt{G_{11}\,G_{22}}\,\sin(\phi-\theta) \\
&= \sqrt{G_{11}\,G_{22}}\,\cos\psi \\
&= (1+\gamma_1+\gamma_2+\gamma_1\gamma_2)^{1/2}\,\cos\psi
\end{aligned} \tag{19}$$

Note from the above that

$$G_{12} = -J\,\tan\psi \tag{20}$$

2.3 Contravariant metrics

We now find the contravariant metrics as defined from equation (2.4)

$$G^{11} = J^{-2}\,G_{22}\,G_{33} = \frac{(1+\gamma_2)}{J^2} \tag{21.a}$$

$$G^{22} = J^{-2}\,G_{11}\,G_{33} = \frac{(1+\gamma_1)}{J^2} \tag{21.b}$$

$$G^{33} = J^{-2}\,G_{11}\,G_{22} = \frac{(1+\gamma_1+\gamma_2+\gamma_1\gamma_2)}{J^2} \tag{21.c}$$

$$\begin{aligned}
G^{12} &= -\,J^{-2}\,G_{12}\,G_{33} \\
&= \frac{\tan\psi}{J}
\end{aligned} \tag{21.d}$$

$$G^{13} = 0 \tag{21.e}$$

$$G^{23} = 0 \tag{21.f}$$

2.4 Christoffel symbols

Covariant differentiation is defined as (Aris, 1962, eq 7.55.4)

$$A^i{}_{,\,j} = \frac{\partial A^i}{\partial \xi^j} + \left\{\begin{matrix} & i & \\ j & & k\end{matrix}\right\} A^k \tag{22}$$

where the Christoffel symbol is (Aris, 1962, eq 7.53.3)

$$\left\{\begin{matrix} & i & \\ j & & k\end{matrix}\right\} = \frac{1}{2}\,G^{ip}\left(\frac{\partial G_{pj}}{\partial \xi^k} + \frac{\partial G_{pk}}{\partial \xi^j} - \frac{\partial G_{jk}}{\partial \xi^p}\right) \tag{23}$$

and it can be seen that

$$\left\{ \begin{matrix} & i & \\ j & & k \end{matrix} \right\} = \left\{ \begin{matrix} & i & \\ k & & j \end{matrix} \right\} \tag{24}$$

As the G^{i3} and G_{i3} metrics are zero for $i \neq 3$, all the Christoffel symbols involving the vertical coordinate (3) mixed with the horizontal components (1) and (2) evaluate to exactly zero by inspection. The non-trivial terms are:

$$\left\{ \begin{matrix} & 1 & \\ 1 & & 1 \end{matrix} \right\} = \frac{1}{2} G^{1p} \left(\frac{\partial G_{p1}}{\partial \xi^1} + \frac{\partial G_{p1}}{\partial \xi^1} - \frac{\partial G_{11}}{\partial \xi^p} \right) \tag{25.a}$$

$$\left\{ \begin{matrix} & 1 & \\ 1 & & 2 \end{matrix} \right\} = \frac{1}{2} G^{1p} \left(\frac{\partial G_{p1}}{\partial \xi^2} + \frac{\partial G_{p2}}{\partial \xi^1} - \frac{\partial G_{12}}{\partial \xi^p} \right) \tag{25.b}$$

$$\left\{ \begin{matrix} & 1 & \\ 2 & & 2 \end{matrix} \right\} = \frac{1}{2} G^{1p} \left(\frac{\partial G_{p2}}{\partial \xi^2} + \frac{\partial G_{p2}}{\partial \xi^2} - \frac{\partial G_{22}}{\partial \xi^p} \right) \tag{25.c}$$

$$\left\{ \begin{matrix} & 2 & \\ 1 & & 1 \end{matrix} \right\} = \frac{1}{2} G^{2p} \left(\frac{\partial G_{p1}}{\partial \xi^1} + \frac{\partial G_{p1}}{\partial \xi^1} - \frac{\partial G_{11}}{\partial \xi^p} \right) \tag{25.d}$$

$$\left\{ \begin{matrix} & 2 & \\ 1 & & 2 \end{matrix} \right\} = \frac{1}{2} G^{2p} \left(\frac{\partial G_{p1}}{\partial \xi^2} + \frac{\partial G_{p2}}{\partial \xi^1} - \frac{\partial G_{12}}{\partial \xi^p} \right) \tag{25.e}$$

$$\left\{ \begin{matrix} & 2 & \\ 2 & & 2 \end{matrix} \right\} = \frac{1}{2} G^{2p} \left(\frac{\partial G_{p2}}{\partial \xi^2} + \frac{\partial G_{p2}}{\partial \xi^2} - \frac{\partial G_{22}}{\partial \xi^p} \right) \tag{25.f}$$

$$\left\{ \begin{matrix} & 3 & \\ 3 & & 3 \end{matrix} \right\} = \frac{1}{2} G^{3p} \left(\frac{\partial G_{p3}}{\partial \xi^3} + \frac{\partial G_{p3}}{\partial \xi^3} - \frac{\partial G_{33}}{\partial \xi^p} \right) \tag{25.g}$$

Evaluating these:

$$\left\{ \begin{matrix} & 1 & \\ 1 & & 1 \end{matrix} \right\} = \frac{1}{2} G^{11} \left(\frac{\partial G_{11}}{\partial \xi^1} \right) + \frac{1}{2} G^{12} \left(2 \frac{\partial G_{12}}{\partial \xi^1} - \frac{\partial G_{11}}{\partial \xi^2} \right) \tag{26.a}$$

$$\left\{ \begin{matrix} & 1 & \\ 1 & & 2 \end{matrix} \right\} = \frac{1}{2} G^{11} \frac{\partial G_{11}}{\partial \xi^2} + \frac{1}{2} G^{12} \frac{\partial G_{22}}{\partial \xi^1} \tag{26.b}$$

$$\left\{ \begin{matrix} & 1 & \\ 2 & & 2 \end{matrix} \right\} = \frac{1}{2} G^{11} \left(2 \frac{\partial G_{12}}{\partial \xi^2} - \frac{\partial G_{22}}{\partial \xi^1} \right) + \frac{1}{2} G^{12} \left(\frac{\partial G_{22}}{\partial \xi^2} \right) \tag{26.c}$$

$$\left\{ \begin{matrix} & 2 & \\ 1 & & 1 \end{matrix} \right\} = \frac{1}{2} G^{21} \frac{\partial G_{11}}{\partial \xi^1} + \frac{1}{2} G^{22} \left(2 \frac{\partial G_{21}}{\partial \xi^1} - \frac{\partial G_{11}}{\partial \xi^2} \right) \tag{26.d}$$

$$\left\{ \begin{matrix} & 2 & \\ 1 & & 2 \end{matrix} \right\} = \frac{1}{2} G^{21} \frac{\partial G_{11}}{\partial \xi^2} + \frac{1}{2} G^{22} \frac{\partial G_{22}}{\partial \xi^1} \tag{26.e}$$

$$\left\{ \begin{matrix} & 2 & \\ 2 & & 2 \end{matrix} \right\} = \frac{1}{2} G^{21} \left(2 \frac{\partial G_{12}}{\partial \xi^2} - \frac{\partial G_{22}}{\partial \xi^1} \right) + \frac{1}{2} G^{22} \frac{\partial G_{22}}{\partial \xi^2} \tag{26.f}$$

$$\left\{ \begin{matrix} & 3 & \\ 3 & & 3 \end{matrix} \right\} = \frac{1}{2} G^{33} \left(\frac{\partial G_{33}}{\partial \xi^3} \right) \tag{26.g}$$

Common terms in the above are evaluated as

$$\frac{\partial G_{11}}{\partial \xi^1} = \frac{\partial \gamma_1}{\partial \xi^1} \tag{27.a}$$

$$\frac{\partial G_{11}}{\partial \xi^2} = \frac{\partial \gamma_1}{\partial \xi^2} \tag{27.b}$$

$$\frac{\partial G_{22}}{\partial \xi^1} = \frac{\partial \gamma_2}{\partial \xi^1} \tag{27.c}$$

$$\frac{\partial G_{22}}{\partial \xi^2} = \frac{\partial \gamma_2}{\partial \xi^2} \tag{27.d}$$

$$\frac{\partial G_{33}}{\partial \xi^3} = 0 \tag{27.e}$$

$$\begin{aligned}
\frac{\partial G_{12}}{\partial \xi^1} &= -\left(G_{11}G_{22}\right)^{1/2} \cos\psi \, \frac{\partial \psi}{\partial \xi^1} - \frac{\frac{\partial}{\partial \xi^1}\left(G_{11}G_{22}\right)}{2\left(G_{11}G_{22}\right)^{1/2}} \sin\psi \\
&= -\left(G_{11}G_{22}\right)^{1/2} \cos\psi \, \frac{\partial \psi}{\partial \xi^1} - \frac{G_{22}\frac{\partial G_{11}}{\partial \xi^1} + G_{11}\frac{\partial G_{22}}{\partial \xi^1}}{2\left(G_{11}G_{22}\right)^{1/2}} \sin\psi \\
&= -J \, \frac{\partial \psi}{\partial \xi^1} - \frac{G_{22}\frac{\partial \gamma_1}{\partial \xi^1} + G_{11}\frac{\partial \gamma_2}{\partial \xi^1}}{2\left(G_{11}G_{22}\right)^{1/2}} \sin\psi \\
&= -J \, \frac{\partial \psi}{\partial \xi^1} - \frac{1}{2}\left\{ \left(\frac{G_{22}}{G_{11}}\right)^{\frac{1}{2}} \frac{\partial \gamma_1}{\partial \xi^1} + \left(\frac{G_{11}}{G_{22}}\right)^{\frac{1}{2}} \frac{\partial \gamma_2}{\partial \xi^1} \right\} \sin\psi \\
&= -J \, \frac{\partial \psi}{\partial \xi^1} - \frac{1}{2}\left\{ \left(\frac{1+\gamma_2}{1+\gamma_1}\right)^{\frac{1}{2}} \frac{\partial \gamma_1}{\partial \xi^1} + \left(\frac{1+\gamma_1}{1+\gamma_2}\right)^{\frac{1}{2}} \frac{\partial \gamma_2}{\partial \xi^1} \right\} \sin\psi
\end{aligned} \tag{27.f}$$

$$\frac{\partial G_{12}}{\partial \xi^2} = -J \, \frac{\partial \psi}{\partial \xi^2} - \frac{1}{2}\left\{ \left(\frac{1+\gamma_1}{1+\gamma_2}\right)^{\frac{1}{2}} \frac{\partial \gamma_2}{\partial \xi^2} + \left(\frac{1+\gamma_2}{1+\gamma_1}\right)^{\frac{1}{2}} \frac{\partial \gamma_1}{\partial \xi^2} \right\} \sin\psi \tag{27.g}$$

For small ψ, we can say $\sin\psi \approx \tan\psi \approx \partial\psi/\partial\xi \approx O\left(\psi\right)$. It follows that $G^{12} \approx G_{12} \approx \partial G_{12}/\partial\psi \approx O\left(\psi\right)$. Then, to order $\left(\psi^2\right)$, substituting for the gradients of contravariant metrics we can write the Christoffel symbols as:

$$\left\{ \begin{matrix} 1 \\ 1 \quad 1 \end{matrix} \right\} = \frac{1}{2} G^{11} \frac{\partial \gamma_1}{\partial \xi^1} - \frac{1}{2} G^{12} \frac{\partial \gamma_1}{\partial \xi^2} + \left(\psi^2\right) \tag{28.a}$$

$$\left\{ \begin{matrix} 1 \\ 1 \quad 2 \end{matrix} \right\} = \frac{1}{2} G^{11} \frac{\partial \gamma_1}{\partial \xi^2} + \frac{1}{2} G^{12} \frac{\partial \gamma_2}{\partial \xi^1} \tag{28.b}$$

$$\begin{aligned}
\left\{ \begin{matrix} 1 \\ 2 \quad 2 \end{matrix} \right\} &= -\frac{1}{2} G^{11} \frac{\partial \gamma_2}{\partial \xi^1} + \frac{1}{2} G^{12} \frac{\partial \gamma_2}{\partial \xi^2} + -J\, G^{11} \frac{\partial \psi}{\partial \xi^2} \\
&\quad - \frac{1}{2} G^{11} \left\{ \left(\frac{1+\gamma_1}{1+\gamma_2}\right)^{\frac{1}{2}} \frac{\partial \gamma_2}{\partial \xi^2} + \left(\frac{1+\gamma_2}{1+\gamma_1}\right)^{\frac{1}{2}} \frac{\partial \gamma_1}{\partial \xi^2} \right\} \sin\psi
\end{aligned} \tag{28.c}$$

$$\begin{aligned}
\left\{ \begin{matrix} 2 \\ 1 \quad 1 \end{matrix} \right\} &= \frac{1}{2} G^{21} \frac{\partial \gamma_1}{\partial \xi^1} - \frac{1}{2} G^{22} \frac{\partial \gamma_1}{\partial \xi^2} - J\, G^{22} \frac{\partial \psi}{\partial \xi^1} \\
&\quad - \frac{1}{2} G^{22} \left\{ \left(\frac{1+\gamma_2}{1+\gamma_1}\right)^{\frac{1}{2}} \frac{\partial \gamma_1}{\partial \xi^1} + \left(\frac{1+\gamma_1}{1+\gamma_2}\right)^{\frac{1}{2}} \frac{\partial \gamma_2}{\partial \xi^1} \right\} \sin\psi
\end{aligned} \tag{28.d}$$

$$\left\{ \begin{matrix} 2 \\ 1 \quad 2 \end{matrix} \right\} = \frac{1}{2} G^{21} \frac{\partial \gamma_1}{\partial \xi^2} + \frac{1}{2} G^{22} \frac{\partial \gamma_2}{\partial \xi^1} \tag{28.e}$$

$$\begin{aligned}
\left\{ \begin{matrix} 2 \\ 2 \quad 2 \end{matrix} \right\} &= \frac{1}{2} G^{22} \frac{\partial \gamma_2}{\partial \xi^2} - \frac{1}{2} G^{21} \frac{\partial \gamma_2}{\partial \xi^1} + O\left(\psi^2\right) \\
\left\{ \begin{matrix} 3 \\ 3 \quad 3 \end{matrix} \right\} &= 0
\end{aligned} \tag{28.f}$$

Substituting the relations for G^{11}, G^{22} and G^{12} provides:

$$\left\{ \begin{matrix} 1 \\ 1 \quad 1 \end{matrix} \right\} = \frac{1+\gamma_2}{2J^2}\frac{\partial \gamma_1}{\partial \xi^1} - \frac{\tan\psi}{2J}\frac{\partial \gamma_1}{\partial \xi^2} + O\left(\psi^2\right) \tag{29.a}$$

$$\left\{ \begin{matrix} 1 \\ 1 \quad 2 \end{matrix} \right\} = \frac{1+\gamma_2}{2J^2}\frac{\partial \gamma_1}{\partial \xi^2} + \frac{\tan\psi}{2J}\frac{\partial \gamma_2}{\partial \xi^1} \tag{29.b}$$

$$\begin{aligned} \left\{ \begin{matrix} 1 \\ 2 \quad 2 \end{matrix} \right\} &= -\frac{1+\gamma_2}{2J^2}\frac{\partial \gamma_2}{\partial \xi^1} + \frac{\tan\psi}{2J}\frac{\partial \gamma_2}{\partial \xi^2} + -\frac{1+\gamma_2}{J}\frac{\partial \psi}{\partial \xi^2} \\ &\quad - \frac{1+\gamma_2}{2J^2}\left\{\left(\frac{1+\gamma_1}{1+\gamma_2}\right)^{\frac{1}{2}}\frac{\partial \gamma_2}{\partial \xi^2} + \left(\frac{1+\gamma_2}{1+\gamma_1}\right)^{\frac{1}{2}}\frac{\partial \gamma_1}{\partial \xi^2}\right\}\sin\psi \end{aligned} \tag{29.c}$$

$$\begin{aligned} \left\{ \begin{matrix} 2 \\ 1 \quad 1 \end{matrix} \right\} &= \frac{\tan\psi}{2J}\frac{\partial \gamma_1}{\partial \xi^1} - \frac{1+\gamma_1}{2J^2}\frac{\partial \gamma_1}{\partial \xi^2} - \frac{1+\gamma_1}{2J}\frac{\partial \psi}{\partial \xi^1} \\ &\quad - \frac{1+\gamma_1}{2J^2}\left\{\left(\frac{1+\gamma_2}{1+\gamma_1}\right)^{\frac{1}{2}}\frac{\partial \gamma_1}{\partial \xi^1} + \left(\frac{1+\gamma_1}{1+\gamma_2}\right)^{\frac{1}{2}}\frac{\partial \gamma_2}{\partial \xi^1}\right\}\sin\psi \end{aligned} \tag{29.d}$$

$$\left\{ \begin{matrix} 2 \\ 1 \quad 2 \end{matrix} \right\} = \frac{1+\gamma_1}{2J^2}\frac{\partial \gamma_2}{\partial \xi^1} + \frac{\tan\psi}{2J}\frac{\partial \gamma_1}{\partial \xi^2} \tag{29.e}$$

$$\left\{ \begin{matrix} 2 \\ 2 \quad 2 \end{matrix} \right\} = \frac{1+\gamma_1}{2J^2}\frac{\partial \gamma_2}{\partial \xi^2} - \frac{\tan\psi}{2J}\frac{\partial \gamma_2}{\partial \xi^1} + O\left(\psi^2\right) \tag{29.f}$$

If we neglect terms of $O\left(\psi\right)$ (i.e. the non-orthogonality of the transformation) then we arrive at the Christoffel symbols

$$\left\{ \begin{matrix} 1 \\ 1 \quad 1 \end{matrix} \right\} = \frac{1+\gamma_2}{2J^2}\frac{\partial \gamma_1}{\partial \xi^1} + O\left(\psi\right) \tag{30.a}$$

$$\left\{ \begin{matrix} 1 \\ 1 \quad 2 \end{matrix} \right\} = \frac{1+\gamma_2}{2J^2}\frac{\partial \gamma_1}{\partial \xi^2} + O\left(\psi\right) \tag{30.b}$$

$$\left\{ \begin{matrix} 1 \\ 2 \quad 2 \end{matrix} \right\} = -\frac{1+\gamma_2}{2J^2}\frac{\partial \gamma_2}{\partial \xi^1} + O\left(\psi\right) \tag{30.c}$$

$$\left\{ \begin{matrix} 2 \\ 1 \quad 1 \end{matrix} \right\} = -\frac{1+\gamma_1}{2J^2}\frac{\partial \gamma_1}{\partial \xi^2} + O\left(\psi\right) \tag{30.d}$$

$$\left\{ \begin{matrix} 2 \\ 1 \quad 2 \end{matrix} \right\} = \frac{1+\gamma_1}{2J^2}\frac{\partial \gamma_2}{\partial \xi^1} + O\left(\psi\right)$$

$$\left\{ \begin{matrix} 2 \\ 2 \quad 2 \end{matrix} \right\} = \frac{1+\gamma_1}{2J^2}\frac{\partial \gamma_2}{\partial \xi^2} + O\left(\psi\right) \tag{30.e}$$

2.5 Covariant velocity derivatives

Neglecting the $O(\psi)$ terms we can write the covariant derivatives of the velocity field as:

$$U^1_{\ ,1} = \frac{\partial U^1}{\partial \xi^1} + \frac{1+\gamma_2}{2J^2}\left[U^1\frac{\partial \gamma_1}{\partial \xi^1} + U^2\frac{\partial \gamma_1}{\partial \xi^2}\right] + O(\psi) \tag{31.a}$$

$$U^2_{,2} = \frac{\partial U^2}{\partial \xi^2} + \frac{1+\gamma_1}{2J^2}\left[U^1\frac{\partial \gamma_2}{\partial \xi^1} + U^2\frac{\partial \gamma_2}{\partial \xi^2}\right] + O(\psi) \quad (31.b)$$

$$U^i_{,3} = \frac{\partial U^i}{\partial \xi^3} \quad (31.c)$$

$$U^3_{,i} = \frac{\partial U^3}{\partial \xi^i} \quad (31.d)$$

$$U^1_{,2} = \frac{\partial U^1}{\partial \xi^2} + \frac{1+\gamma_2}{2J^2}\left[U^1\frac{\partial \gamma_1}{\partial \xi^2} - U^2\frac{\partial \gamma_2}{\partial \xi^1}\right] + O(\psi) \quad (31.e)$$

$$U^2_{,1} = \frac{\partial U^2}{\partial \xi^1} + \frac{1+\gamma 1}{2J^2}\left[U^2\frac{\partial \gamma_2}{\partial \xi^1} - U^1\frac{\partial \gamma_1}{\partial \xi^2}\right] + O(\psi) \quad (31.f)$$

2.6 Terms in the Navier-Stokes equations

2.6.1 Advective terms

Examining the advective terms for the (1) component of the Navier-Stokes equations:

$$\begin{aligned} U^1U^1_{,1} + U^2U^1_{,2} + U^3U^1_{,3} \\ = U^1\left\{\frac{\partial U^1}{\partial \xi^1} + \frac{1+\gamma_2}{2J^2}\left[U^1\frac{\partial \gamma_1}{\partial \xi^1} + U^2\frac{\partial \gamma_1}{\partial \xi^2}\right]\right\} \\ + U^2\left\{\frac{\partial U^1}{\partial \xi^2} + \frac{1+\gamma_2}{2J^2}\left[U^1\frac{\partial \gamma_1}{\partial \xi^2} - U^2\frac{\partial \gamma_2}{\partial \xi^1}\right]\right\} \\ + U^3\frac{\partial U^1}{\partial \xi^3} + O(\psi) \end{aligned} \quad (32)$$

This can be written as

$$\begin{aligned} U^1U^1_{,1} + U^2U^1_{,2} + U^3U^1_{,3} \\ = U^1\frac{\partial U^1}{\partial \xi^1} + U^2\frac{\partial U^1}{\partial \xi^2} + U^3\frac{\partial U^1}{\partial \xi^3} \\ + \frac{(1+\gamma_2)}{2J^2}\left\{\frac{\partial \gamma_1}{\partial \xi^1}\left(U^1\right)^2 + 2\frac{\partial \gamma_1}{\partial \xi^2}U^1U^2 - \frac{\partial \gamma_2}{\partial \xi^1}\left(U^2\right)^2\right\} + O(\psi) \end{aligned} \quad (33)$$

Substituting $J^2 = (1+\gamma_2)(1+\gamma_1) + O(\psi)$

$$\begin{aligned} U^1U^1_{,1} + U^2U^1_{,2} + U^3U^1_{,3} \\ = U^1\frac{\partial U^1}{\partial \xi^1} + U^2\frac{\partial U^1}{\partial \xi^2} + U^3\frac{\partial U^1}{\partial \xi^3} \\ + \frac{1}{2(1+\gamma_1)}\left\{\frac{\partial \gamma_1}{\partial \xi^1}\left(U^1\right)^2 + 2\frac{\partial \gamma_1}{\partial \xi^2}U^1U^2 - \frac{\partial \gamma_2}{\partial \xi^1}\left(U^2\right)^2\right\} + O(\psi) \end{aligned} \quad (34)$$

2.6.2 Free-surface terms

Next consider the first type of free surface term from the N-S equations. For the (1) component

$$
\begin{aligned}
g\,G^{1\alpha}\frac{\partial H}{\partial \xi^\alpha} &= g\,G^{11}\frac{\partial H}{\partial \xi^1} + g\,G^{12}\frac{\partial H}{\partial \xi^2} \\
&= g\,\frac{(1+\gamma_2)}{J^2}\frac{\partial H}{\partial \xi^1} + g\,\frac{1}{J}\tan\psi\,\frac{\partial H}{\partial \xi^2} \\
&= g\,\frac{\partial H}{\partial \xi^1} + g\left(\frac{1+\gamma_2}{J^2} - 1\right)\frac{\partial H}{\partial \xi^1} + g\,\frac{1}{J}\tan\psi\,\frac{\partial H}{\partial \xi^2} \\
&= g\,\frac{\partial H}{\partial \xi^1} + \frac{g}{J^2}\left(1 - J^2 + \gamma_2\right)\frac{\partial H}{\partial \xi^1} + \frac{g}{J}\tan\psi\,\frac{\partial H}{\partial \xi^2} \\
&= g\,\frac{\partial H}{\partial \xi^1} + \frac{g}{J^2}\left(1 - J^2 + \gamma_2\right)\frac{\partial H}{\partial \xi^1} + O(\psi) \\
&= g\,\frac{\partial H}{\partial \xi^1} + \frac{g\,[1+\gamma_2 - (1+\gamma_1)(1+\gamma_2)]}{(1+\gamma_1)(1+\gamma_2)}\frac{\partial H}{\partial \xi^1} + O(\psi) \\
&= g\,\frac{\partial H}{\partial \xi^1} - \frac{g\,\gamma_1\,(1+\gamma_2)}{(1+\gamma_1)(1+\gamma_2)}\frac{\partial H}{\partial \xi^1} + O(\psi) \\
&= g\,\frac{\partial H}{\partial \xi^1} - \frac{g\,\gamma_1}{1+\gamma_1}\frac{\partial H}{\partial \xi^1} + O(\psi) \qquad (35)
\end{aligned}
$$

2.6.3 Baroclinic terms

The baroclinic term in the Navier-Stokes equations for the (1) component is

$$
\begin{aligned}
&\frac{g\,G^{1\alpha}}{\rho_0}\frac{\partial}{\partial \xi^\alpha}\int_{z'}^{H}\rho'\,dz \\
&= \frac{g\,G^{11}}{\rho_0}\frac{\partial}{\partial \xi^1}\int_{z'}^{H}\rho'\,dz + \frac{g\,G^{12}}{\rho_0}\frac{\partial}{\partial \xi^2}\int_{z'}^{H}\rho'\,dz \\
&= \frac{g\,(1+\gamma_2)}{J^2\,\rho_0}\frac{\partial}{\partial \xi^1}\int_{z'}^{H}\rho'\,dz + \frac{g\,\tan\psi}{J\,\rho_0}\frac{\partial}{\partial \xi^2}\int_{z'}^{H}\rho'\,dz \\
&= \frac{g}{\rho_0}\frac{\partial}{\partial \xi^1}\int_{z'}^{H}\rho'\,dz + \frac{g\left(1+\gamma_2 - J^2\right)}{J^2\,\rho_0}\frac{\partial}{\partial \xi^1}\int_{z'}^{H}\rho'\,dz + \frac{g\,\tan\psi}{J\,\rho_0}\frac{\partial}{\partial \xi^2}\int_{z'}^{H}\rho'\,dz \\
&= \frac{g}{\rho_0}\frac{\partial}{\partial \xi^1}\int_{z'}^{H}\rho'\,dz + \frac{g\,(1+\gamma_2 - [1+\gamma_1][1+\gamma_2])}{\rho_0\,[1+\gamma_1][1+\gamma_2]}\frac{\partial}{\partial \xi^1}\int_{z'}^{H}\rho'\,dz + O(\psi) \\
&= \frac{g}{\rho_0}\frac{\partial}{\partial \xi^1}\int_{z'}^{H}\rho'\,dz - \frac{g\,\gamma_1}{\rho_0\,(1+\gamma_1)}\frac{\partial}{\partial \xi^1}\int_{z'}^{H}\rho'\,dz + O(\psi) \qquad (36)
\end{aligned}
$$

2.7 Grid-stretched form for N-S equations

Putting together the previous terms, we arrive at a statement of the Navier-Stokes equations for the (1) and (2) components:

$$\begin{aligned}\frac{\partial U^1}{\partial t} &+ U^j \frac{\partial U^1}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^1} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz - \nu U^1_{,kk} \\ &= -\frac{1}{2(1+\gamma_1)} \left\{ \frac{\partial \gamma_1}{\partial \xi^1} \left(U^1\right)^2 + 2 \frac{\partial \gamma_1}{\partial \xi^2} U^1 U^2 - \frac{\partial \gamma_2}{\partial \xi^1} \left(U^2\right)^2 \right\} \\ &\quad + \frac{g\,\gamma_1}{1+\gamma_1} \left\{ \frac{\partial H}{\partial \xi^1} + \frac{1}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho'\, dz \right\} + O(\psi) \end{aligned} \qquad (37.a)$$

$$\begin{aligned}\frac{\partial U^2}{\partial t} &+ U^j \frac{\partial U^2}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^2} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^2} \int_{z'}^{H} \rho' dz - \nu U^2_{,kk} \\ &= -\frac{1}{2(1+\gamma_2)} \left\{ \frac{\partial \gamma_2}{\partial \xi^2} \left(U^2\right)^2 + 2 \frac{\partial \gamma_2}{\partial \xi^1} U^1 U^2 - \frac{\partial \gamma_1}{\partial \xi^2} \left(U^1\right)^2 \right\} \\ &\quad + \frac{g\,\gamma_2}{1+\gamma_2} \left\{ \frac{\partial H}{\partial \xi^2} + \frac{1}{\rho_0} \frac{\partial}{\partial \xi^2} \int_{z'}^{H} \rho'\, dz \right\} + O(\psi) \end{aligned} \qquad (37.b)$$

Where the curvilinear grid stretches in only one direction, i.e. where $\gamma_2 = 0$ and $\psi = 0$ we obtain

$$\begin{aligned}\frac{\partial U^1}{\partial t} &+ U^j \frac{\partial U^1}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^1} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz - \nu U^1_{,kk} \\ &= -\frac{1}{2(1+\gamma_1)} \left\{ \frac{\partial \gamma_1}{\partial \xi^1} \left(U^1\right)^2 + 2 \frac{\partial \gamma_1}{\partial \xi^2} U^1 U^2 \right\} \\ &\quad + \frac{g\,\gamma_1}{(1+\gamma_1)} \left\{ \frac{\partial H}{\partial \xi^1} + \frac{1}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz \right\} + O(\psi) \end{aligned} \qquad (38.a)$$

$$\begin{aligned}\frac{\partial U^2}{\partial t} &+ U^j \frac{\partial U^2}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^2} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^2} \left\{ \int_{z'}^{H} \rho' dz \right\} - \nu U^2_{,kk} \\ &= +\frac{1}{2} \frac{\partial \gamma_1}{\partial \xi^2} \left(U^1\right)^2 + O(\psi) \end{aligned} \qquad (38.b)$$

It is useful to consider the case where there stretching is only in one direction and only changes across the second dimension, i.e. $\gamma_2 = 0$ and $\partial \gamma_1 / \partial \xi^1 = 0$:

$$\begin{aligned}\frac{\partial U^1}{\partial t} &+ U^j \frac{\partial U^1}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^1} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz - \nu U^1_{,kk} \\ &= -\frac{1}{(1+\gamma_1)} \frac{\partial \gamma_1}{\partial \xi^2} U^1 U^2 + \frac{g\,\gamma_1}{(1+\gamma_1)} \left\{ \frac{\partial H}{\partial \xi^1} + \frac{1}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz \right\} + O(\psi) \end{aligned}$$

(39.a)

$$\frac{\partial U^2}{\partial t} \;+\; U^j \frac{\partial U^2}{\partial \xi^j} \;+\; g \frac{\partial H}{\partial \xi^2} \;+\; \frac{g}{\rho_0} \frac{\partial}{\partial \xi^2} \left\{ \int_{z'}^{H} \rho' dz \right\} \;-\; \nu \, U^2_{,kk}$$
$$= \; + \frac{1}{2} \frac{\partial \gamma_1}{\partial \xi^2} \left(U^1 \right)^2 \;+\; O(\psi) \qquad (39.b)$$

2.8 Correspondence with cylindrical polar coordinates

As a demonstration of the validity of the curvilinear form, consider a curvilinear space that may be defined by the cylindrical polar coordinates (r, θ, z). The correspondence between the curvilinear form and the polar form is

$$\xi^1 = r_c \, \theta \qquad (40.a)$$
$$\xi^2 = r \qquad (40.b)$$
$$\xi^3 = z \qquad (40.c)$$

where r_c is some "central" radius of curvature. The cylindrical polar system is orthogonal so $\psi = 0$. The velocities transform as

$$U^1 = \frac{r_c}{r} \, u_\theta \qquad (41.a)$$
$$U^2 = u_r \qquad (41.b)$$
$$U^3 = u_z \qquad (41.c)$$

Transforming the derivatives, we have

$$\frac{\partial}{\partial \xi^1} = \frac{\partial \theta}{\partial \xi^1} \frac{\partial}{\partial \theta} = \frac{1}{r_c} \frac{\partial}{\partial \theta} \qquad (42.a)$$
$$\frac{\partial}{\partial \xi^2} = \frac{\partial r}{\partial \xi^2} \frac{\partial}{\partial r} = \frac{\partial}{\partial r} \qquad (42.b)$$
$$\frac{\partial}{\partial \xi^3} = \frac{\partial z}{\partial \xi^3} \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \qquad (42.c)$$

For a sufficiently small θ where the arc is aligned with the Cartesian x axis we have $\Delta x = r \, \Delta\theta$ so we find

$$G_{11} = \frac{\Delta x}{\Delta \xi^1} \frac{\Delta x}{\Delta \xi^1} = \frac{r \, \Delta\theta}{r_c \, \Delta\theta} \frac{r \Delta\theta}{r_c \Delta\theta} = 1 \;+\; \gamma_1 \qquad (43)$$

As the cylindrical polar coordinate system is uniform in the θ direction, this result holds throughout the curvilinear system. It follows that:

$$\gamma_1 = \frac{r^2}{r_c^2} - 1 = \frac{r^2 \, - \, r_c^2}{r_c^2} \qquad (44.a)$$
$$\gamma_2 = 0 \qquad (44.b)$$
$$\psi = 0 \qquad (44.c)$$
$$J^2 = 1 + \gamma_1 = \frac{r^2}{r_c^2} \qquad (44.d)$$

$$\frac{\partial \gamma_1}{\partial \xi_1} = 0 \tag{44.e}$$

$$\frac{\partial \gamma_1}{\partial \xi_2} = \frac{\partial \gamma_1}{\partial r} = \frac{2\,r}{r_c^2} \tag{44.f}$$

$$\frac{\partial^2 \gamma_1}{\partial \xi_2 \partial \xi_2} = \frac{\partial^2 \gamma_1}{\partial r^2} = \frac{2}{r_c^2} \tag{44.g}$$

Substituting the polar coordinate relations into the Navier-Stokes equations (39.a) and (39.b) for the conditions $\partial\gamma_1/\partial\xi^1 = 0$ with $\psi = 0$ and $\gamma_2 = 0$ we obtain

$$\begin{aligned}
&\frac{r_c}{r}\frac{\partial u_\theta}{\partial t} + \frac{1}{r_c}\left(\frac{r_c}{r}u_\theta\right)\frac{\partial}{\partial\theta}\left(\frac{r_c}{r}u_\theta\right) + u_r\frac{\partial}{\partial r}\left(\frac{r_c}{r}u_\theta\right) + u_z\frac{\partial}{\partial z}\left(\frac{r_c}{r}u_\theta\right) + g\,\frac{1}{r_c}\frac{\partial H}{\partial\theta} \\
&\quad + \frac{g}{\rho_0}\frac{1}{r_c}\frac{\partial}{\partial\theta}\int_{z'}^{H}\rho' dz \\
&\quad = -\left(\frac{r_c^2}{r^2}\right)\left(\frac{2r}{r_c^2}\right)\left(\frac{r_c}{r}\right)u_\theta\,u_r + g\left(\frac{r^2-r_c^2}{r_c^2}\right)\left(\frac{r_c^2}{r^2}\right)\left\{\frac{1}{r_c}\frac{\partial H}{\partial\theta} + \frac{1}{\rho_0\,r_c}\frac{\partial}{\partial\theta}\int_{z'}^{H}\rho' dz\right\} \\
&\qquad + \text{viscous terms}
\end{aligned} \tag{45.a}$$

$$\begin{aligned}
&\frac{\partial u_r}{\partial t} + \frac{1}{r_c}\left(\frac{r_c}{r}u_\theta\right)\frac{\partial u_r}{\partial\theta} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} + g\,\frac{\partial H}{\partial r} + \frac{g}{\rho_0}\frac{\partial}{\partial r}\int_{z'}^{H}\rho' dz \\
&\quad = \frac{1}{2}\frac{2\,r}{r_c^2}\left(\frac{r_c}{r}u_\theta\right)^2 + \text{viscous terms}
\end{aligned} \tag{45.b}$$

For simplicity, we are not transforming the viscous terms. Note that the derivatives with respect to the radial direction of the U^1 velocity become

$$\frac{\partial}{\partial r}\left(\frac{r_c}{r}u_\theta\right) = \frac{r_c}{r}\frac{\partial u_\theta}{\partial r} - u_\theta\,\frac{r_c}{r^2} \tag{46}$$

while the second derivatives expand as

$$\begin{aligned}
\frac{\partial^2}{\partial r^2}\left(\frac{r_c}{r}u_\theta\right) &= \frac{\partial}{\partial r}\left(\frac{r_c}{r}\frac{\partial u_\theta}{\partial r} - u_\theta\,\frac{r_c}{r^2}\right) \\
&= \frac{r_c}{r}\frac{\partial^2 u_\theta}{\partial r^2} - \frac{r_c}{r^2}\frac{\partial u_\theta}{\partial r} + 2\,u_\theta\,\frac{r_c}{r^3} - \frac{r_c}{r^2}\frac{\partial u_\theta}{\partial r} \\
&= \frac{r_c}{r}\frac{\partial^2 u_\theta}{\partial r^2} - \frac{2\,r_c}{r^2}\frac{\partial u_\theta}{\partial r} + 2\,u_\theta\,\frac{r_c}{r^3}
\end{aligned} \tag{47}$$

Cancelling terms and substituting the derivative expansions provides

$$\begin{aligned}
&\frac{r_c}{r}\frac{\partial u_\theta}{\partial t} + \left(\frac{r_c}{r^2}u_\theta\right)\frac{\partial u_\theta}{\partial\theta} + \frac{r_c}{r}u_r\frac{\partial u_\theta}{\partial r} - u_r\,u_\theta\,\frac{r_c}{r^2} + \frac{r_c}{r}u_z\frac{\partial u_\theta}{\partial z} + g\,\frac{1}{r_c}\frac{\partial H}{\partial\theta} \\
&\quad + \frac{g}{\rho_0}\frac{1}{r_c}\frac{\partial}{\partial\theta}\int_{z'}^{H}\rho' dz \\
&\quad = -\left(\frac{2\,r_c}{r^2}\right)u_\theta\,u_r + g\left(\frac{r^2-r_c^2}{r^2}\right)\left\{\frac{1}{r_c}\frac{\partial H}{\partial\theta} + \frac{1}{\rho_0\,r_c}\frac{\partial}{\partial\theta}\int_{z'}^{H}\rho' dz\right\}
\end{aligned}$$

$$+ \text{ viscous terms} \qquad (48.a)$$

$$\frac{\partial u_r}{\partial t} + \frac{1}{r} u_\theta \frac{\partial u_r}{\partial \theta} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} + g \frac{\partial H}{\partial r} + \frac{g}{\rho_0} \frac{\partial}{\partial r} \int_{z'}^{H} \rho' dz$$
$$= \frac{1}{r} u_\theta^2 + \text{ viscous terms} \qquad (48.b)$$

Multiply the first equation through by r/r_c and clean up:

$$\frac{\partial u_\theta}{\partial t} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_r \frac{\partial u_\theta}{\partial r} - \frac{u_r\, u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z} + g \frac{r}{r_c^2} \frac{\partial H}{\partial \theta}$$
$$+ \frac{g}{\rho_0} \frac{r}{r_c^2} \frac{\partial}{\partial \theta} \int_{z'}^{H} \rho' dz$$
$$= - \frac{2\, u_\theta\, u_r}{r_c} + g \left(\frac{r}{r_c^2} - \frac{1}{r} \right) \left\{ \frac{\partial H}{\partial \theta} + \frac{1}{\rho_0} \frac{\partial}{\partial \theta} \int_{z'}^{H} \rho' dz \right\}$$
$$+ \text{ viscous terms} \qquad (49.a)$$

$$\frac{\partial u_r}{\partial t} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} + g \frac{\partial H}{\partial r} + \frac{g}{\rho_0} \frac{\partial}{\partial r} \int_{z'}^{H} \rho' dz$$
$$= \frac{u_\theta^2}{r} + \text{ viscous terms} \qquad (49.b)$$

Some reworking makes a cancellations more obvious

$$\frac{\partial u_\theta}{\partial t} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_r \frac{\partial u_\theta}{\partial r} - \frac{u_r\, u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z} + g \frac{r}{r_c^2} \frac{\partial H}{\partial \theta} + \frac{g}{\rho_0} \frac{r}{r_c^2} \frac{\partial}{\partial \theta} \int_{z'}^{H} \rho' dz$$
$$= - \frac{2\, u_\theta\, u_r}{r} + g \left(1 - \frac{r_c^2}{r^2} \right) \left\{ \frac{r}{r_c^2} \frac{\partial H}{\partial \theta} + \frac{1}{\rho_0} \frac{r}{r_c^2} \frac{\partial}{\partial \theta} \int_{z'}^{H} \rho' dz \right\} + \text{ viscous term}$$
$$(50.a)$$

$$\frac{\partial u_r}{\partial t} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} + g \frac{\partial H}{\partial r} + \frac{g}{\rho_0} \frac{\partial}{\partial r} \int_{z'}^{H} \rho' dz$$
$$= \frac{(u_\theta)^2}{r} + \text{ viscous term} \qquad (50.b)$$

Further combining of terms and cleaning up provides

$$\frac{\partial u_\theta}{\partial t} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial u_\theta}{\partial z}$$
$$= - \frac{u_\theta\, u_r}{r} - \frac{g}{r} \frac{\partial H}{\partial \theta} - \frac{g}{r\, \rho_0} \frac{\partial}{\partial \theta} \int_{z'}^{H} \rho' dz + \text{ viscous term} \qquad (51.a)$$

$$\frac{\partial u_r}{\partial t} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} + g \frac{\partial H}{\partial r} + \frac{g}{\rho_0} \frac{\partial}{\partial r} \int_{z'}^{H} \rho' dz$$
$$= \frac{(u_\theta)^2}{r} + \text{ viscous term} \qquad (51.b)$$

Any fluid mechanics textbook can be consulted to confirm that this is the cylindrical polar coordinate form of the Navier-Stokes equations with the Bousinesq and hydrostatic pressure approximations. Thus, we have confirmed that the curvilinear form we derived simplifies appropriately when transformed back to physical space.

Chapter 3

Perturbation expansion for a river or estuary

Let us define a "river" curvilinear coordinate system where the ξ^2 coordinate varies along straight lines in physical space that intersect at right angles a set of smoothly curving lines along which ξ^1 varies. One value of constant ξ^2 (i.e. a ξ^1 line) is designated as the "central" coordinate line which has the central radius of curvature $r_c\big(\xi^1\big)$. We require distances along straight lines of the ξ^2 coordinate to be physical distances so that $\Delta\xi^2 = \sqrt{\Delta x^2 + \Delta y^2}$. We also requre that distances measured along the central arc of ξ^1 to be physical distances such that $\Delta\xi^1|_{r_c} = \sqrt{\Delta x^2 + \Delta y^2}$. To prevent overlapping grid points, it is necessary to require that the local radius of curvature $r\big(\xi^1, \xi^2\big)$ is colinear with and a small perturbation from the central radius of curvature at the same value of ξ^1. The ξ^1 coordinate is similar to θ in the cylindrical polar coordinate system except that the central radius of curvature (r_c) changes as a function of ξ^1. The limitation of the river system to values of r that are small perturbations from r_c can be formalized by defining a perturbation parameter ϵ such that

$$\epsilon \equiv \frac{r - r_c}{r_c} \tag{1.a}$$

$$|\epsilon| \ll 1 \tag{1.b}$$

The normal distance of any point from the central radius r_c is defined as

$$\delta\left(\xi^2\right) \equiv r\left(\xi^1, \xi^2\right) - r_c\left(\xi^1\right) \tag{2}$$

which is invariant along a line of varying ξ_1. It follows that

$$\epsilon = \frac{\delta}{r_c} \tag{3}$$

and that

$$\frac{\partial \epsilon}{\partial \xi^1} = -\frac{\delta}{r_c^2}\frac{\partial r_c}{\partial \xi^1} = -\frac{\epsilon}{r_c}\frac{\partial r_c}{\partial \xi^1} = -\frac{\epsilon\,\lambda}{r_c} \tag{4.a}$$

$$\frac{\partial \epsilon}{\partial \xi^2} = \frac{1}{r_c}\frac{\partial r}{\partial \xi^2} = \frac{1}{r_c} \tag{4.b}$$

Where we have defined $\lambda \equiv \partial r_c/\partial \xi^1$ as the gradient of the river curvature. In the same manner as equation (44.a) was obtained, the stretching of the river coordinate system can be defined as:

$$\gamma_1 = \frac{r^2 - r_c^2}{r_c^2} \tag{5.a}$$

$$\gamma_2 \;=\; 0 \tag{5.b}$$

The former can be written as

$$\begin{aligned}
\gamma_1 &= \frac{(r - r_c)\,(r + r_c)}{r_c^2} \\
\gamma_1 &= \frac{\epsilon\,(r + r_c)}{r_c} \\
\gamma_1 &= \epsilon\left(\frac{r}{r_c} + 1\right) \\
\gamma_1 &= \epsilon\,(\epsilon + 2) \\
\gamma_1 &= 2\,\epsilon + \epsilon^2
\end{aligned} \tag{6}$$

It follows that

$$\begin{aligned}
\frac{\partial \gamma_1}{\partial \xi^1} &= -\frac{2\,\epsilon\,\lambda}{r_c} - \frac{2\,\epsilon^2\,\lambda}{r_c} \\
&= -\frac{2\,\epsilon\,\lambda}{r_c}\,(1 + \epsilon)
\end{aligned} \tag{7.a}$$

$$\begin{aligned}
\frac{\partial \gamma_1}{\partial \xi^2} &= \frac{2}{r_c} + \frac{2\,\epsilon}{r_c} \\
&= \frac{2}{r_c}\,(1 + \epsilon)
\end{aligned} \tag{7.b}$$

The primary difference between the cylindrical polar coordinates and the curvilinear system defined herein is that the $\partial \gamma_1 / \partial \xi^1$ is identically zero in the cylindrical polar system, while in the present approach it is a function of the gradient of the radius of curvature $\partial r_c / \partial \xi^1$.

In the continuous sense, the grid is orthogonal as the radii of curvature (lines of varying ξ^2 and constant ξ^1) cut through the lines of varying ξ^1 and constant ξ^2 at angles of $\pi/2$, which results in equations (38.a) and (38.b) being applicable. Substituting the relations for γ into equations (38.a) and (38.b) results in

$$\begin{aligned}
&\frac{\partial U^1}{\partial t} + U^j\,\frac{\partial U^1}{\partial \xi^j} + g\,\frac{\partial H}{\partial \xi^1} + \frac{g}{\rho_0}\,\frac{\partial}{\partial \xi^1}\int_{z'}^{H} \rho' dz - \nu\, U^1_{,kk} \\
&\quad = -\frac{1}{2\,(1 + 2\epsilon + \epsilon^2)}\left\{ -\frac{2\,\epsilon\,\lambda}{r_c}\,(1+\epsilon)\,\left(U^1\right)^2 + 2\,\frac{2}{r_c}\,(1+\epsilon)\,U^1\,U^2 \right\} \\
&\qquad + \frac{g\,\left(2\,\epsilon + \epsilon^2\right)}{(1 + 2\,\epsilon + \epsilon^2)}\left\{ \frac{\partial H}{\partial \xi^1} + \frac{1}{\rho_0}\,\frac{\partial}{\partial \xi^1}\int_{z'}^{H} \rho' dz \right\} + O(\psi)
\end{aligned} \tag{8.a}$$

$$\begin{aligned}
&\frac{\partial U^2}{\partial t} + U^j\,\frac{\partial U^2}{\partial \xi^j} + g\,\frac{\partial H}{\partial \xi^2} + \frac{g}{\rho_0}\,\frac{\partial}{\partial \xi^2}\int_{z'}^{H} \rho' dz - \nu\, U^2_{,kk} \\
&\quad = +\frac{1}{r_c}\,(1+\epsilon)\,\left(U^1\right)^2 + O(\psi)
\end{aligned} \tag{8.b}$$

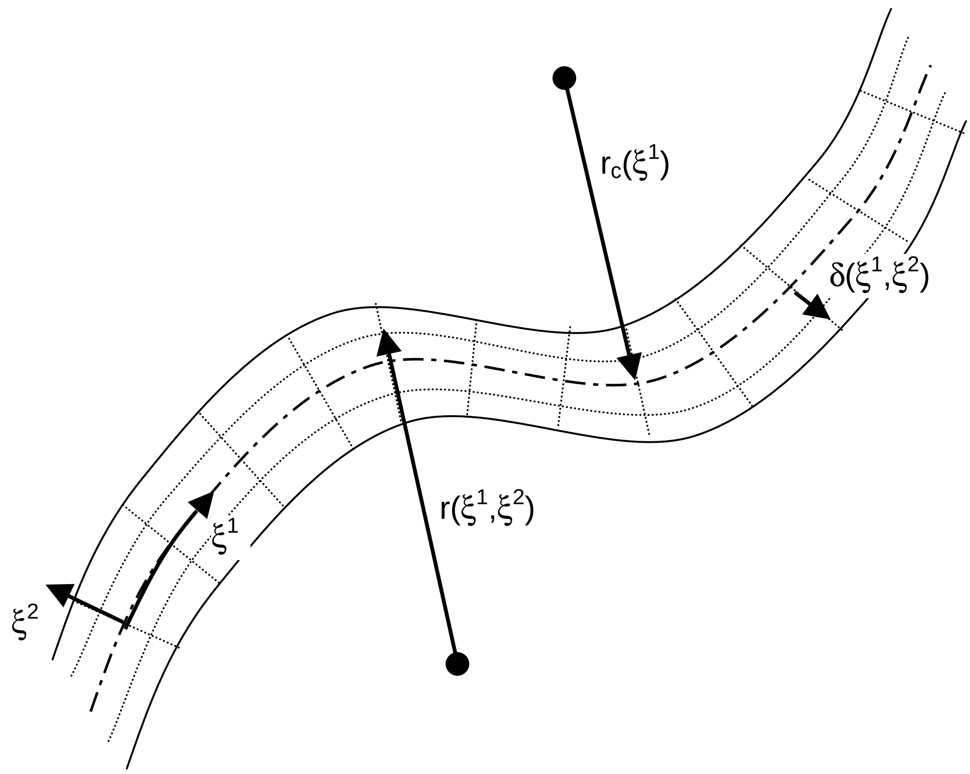

Figure 3.1: Definitions of the central radius of curvature (r_c), the radius of curvature (r) and the distance to the thalweg (δ).

Gathering and cancelling some terms provides

$$\frac{\partial U^1}{\partial t} + U^j \frac{\partial U^1}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^1} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz - \nu U^1_{,kk}$$

$$= - \frac{1}{r_c \, (1+\epsilon)^2} \left\{ - \epsilon \, (1 + \epsilon) \, \lambda \, \left(U^1\right)^2 + 2 \, (1 + \epsilon) \, U^1 U^2 \right\}$$

$$+ \frac{g \, \epsilon \, (2 + \epsilon)}{(1+\epsilon)^2} \left\{ \frac{\partial H}{\partial \xi^1} + \frac{1}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz \right\}$$

$$\frac{\partial U^2}{\partial t} + U^j \frac{\partial U^2}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^2} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^2} \int_{z'}^{H} \rho' dz - \nu U^2_{,kk} = \frac{(1+\epsilon)}{r_c} \left(U^1\right)^2 \qquad (9.a)$$

Further cancellation provides

$$\frac{\partial U^1}{\partial t} + U^j \frac{\partial U^1}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^1} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz - \nu U^1_{,kk}$$

$$= - \frac{1}{r_c \, (1+\epsilon)} \left\{ - \epsilon \, \lambda \, \left(U^1\right)^2 + 2 \, U^1 U^2 \right\}$$

$$+ \frac{g \, \epsilon \, (2 + \epsilon)}{(1+\epsilon)^2} \left\{ \frac{\partial H}{\partial \xi^1} + \frac{1}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz \right\} \qquad (10.a)$$

$$\frac{\partial U^2}{\partial t} + U^j \frac{\partial U^2}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^2} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^2} \int_{z'}^{H} \rho' dz - \nu U^2_{,kk} = \frac{(1+\epsilon)}{r_c} \left(U^1\right)^2 \qquad (10.b)$$

Binomial series expansion (for for $\epsilon^2 < 1$) gives

$$\frac{1}{1+\epsilon} = 1 - \epsilon + \epsilon^2 - O(\epsilon^3) \qquad (11)$$

$$\frac{1}{(1+\epsilon)^2} = 1 - 2\epsilon + 3\epsilon^2 - O(\epsilon^3) \qquad (12)$$

$$(13)$$

So that the equations can be expanded as

$$\frac{\partial U^1}{\partial t} + U^j \frac{\partial U^1}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^1} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz - \nu U^2_{,kk}$$

$$= - \frac{1}{r_c} \left\{ - \epsilon \, \lambda \, \left(U^1\right)^2 + 2 \, U^1 U^2 \right\} + \frac{\epsilon}{r_c} \left\{ - \epsilon \, \lambda \, \left(U^1\right)^2 + 2 \, U^1 U^2 \right\}$$

$$- \frac{\epsilon^2}{r_c} \left\{ - \epsilon \, \lambda \, \left(U^1\right)^2 + 2 \, U^1 U^2 \right\}$$

$$+ g \, \epsilon \, (2 + \epsilon) \left\{ \frac{\partial H}{\partial \xi^1} + \frac{1}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz \right\}$$

$$- 2 \, g \, \epsilon^2 \, (2 + \epsilon) \left\{ \frac{\partial H}{\partial \xi^1} + \frac{1}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz \right\} + O(\epsilon^3)$$

(14.a)

$$\frac{\partial U^2}{\partial t} + U^j \frac{\partial U^2}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^2} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^2} \int_{z'}^{H} \rho' dz - \nu U^2_{,kk} = + \frac{1}{r_c} \left(U^1\right)^2 + \frac{\epsilon}{r_c} \left(U^1\right)^2 \quad (14.b)$$

Regrouping terms

$$\begin{aligned} \frac{\partial U^1}{\partial t} &+ U^j \frac{\partial U^1}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^1} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz - \nu U^1_{,kk} \\ &= + 2 g \epsilon \left\{ \frac{\partial H}{\partial \xi^1} + \frac{1}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz \right\} - \frac{2}{r_c} U^1 U^2 \\ &\quad + \frac{\epsilon}{r_c} \left\{ \lambda \left(U^1\right)^2 + 2 U^1 U^2 \right\} - 2 g \epsilon^2 \left\{ \frac{\partial H}{\partial \xi^1} + \frac{1}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz \right\} \\ &\quad - \frac{\epsilon^2}{r_c} \left\{ \lambda \left(U^1\right)^2 + 2 U^1 U^2 \right\} + O\left(\epsilon^3\right) \end{aligned} \quad (15.a)$$

$$\frac{\partial U^2}{\partial t} + U^j \frac{\partial U^2}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^2} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^2} \int_{z'}^{H} \rho' dz - \nu U^2_{,kk} = + \frac{1}{r_c} \left(U^1\right)^2 + \frac{\epsilon}{r_c} \left(U^1\right)^2 \quad (15.b)$$

For $r_c \gg 1$ and $r_c > \delta$ we can say $\epsilon / r_c \sim O\left(\epsilon^2\right)$ and $\epsilon^2 / r_c \sim O\left(\epsilon^3\right)$ as long as a product of λ is not involved. In general, λ may be $O\left(r_c\right)$. Thus we can write the approximation to order (ϵ^2) as

$$\frac{\partial U^1}{\partial t} + U^j \frac{\partial U^1}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^1} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz - \nu U^1_{,kk} = 2 g \epsilon \left\{ \frac{\partial H}{\partial \xi^1} + \frac{1}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz \right\} - \frac{2}{r_c} U^1 U^2 + \frac{\epsilon \lambda}{r_c} \left(U^1\right)^2 + O\left(\epsilon^2\right) \quad (16.a)$$

$$\frac{\partial U^2}{\partial t} + U^j \frac{\partial U^2}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^2} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^2} \int_{z'}^{H} \rho' dz - \nu U^2_{,kk} = \frac{1}{r_c} \left(U^1\right)^2 + O\left(\epsilon^2\right) \quad (16.b)$$

Finally, for $r_c \gg 1$ and $r_c > \delta$ we can say $r_c^{-1} \sim O(\epsilon)$ so that:

$$\frac{\partial U^1}{\partial t} + U^j \frac{\partial U^1}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^1} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^1} \int_{z'}^{H} \rho' dz - \nu U^1_{,kk} = 0 + O(\epsilon) \quad (17.a)$$

$$\frac{\partial U^2}{\partial t} + U^j \frac{\partial U^2}{\partial \xi^j} + g \frac{\partial H}{\partial \xi^2} + \frac{g}{\rho_0} \frac{\partial}{\partial \xi^2} \int_{z'}^{H} \rho' dz - \nu U^2_{,kk} = 0 + O(\epsilon) \quad (17.b)$$

Chapter 4

Viscous terms

Thus far we have neglected the viscous terms, carrying them along in a curvilinear form. Rather than performing a transformation (requiring derivatives of Christoffel symbols) we can simply neglect molecular viscous processes as they are dominated by turbulent processes throughout our geophysical applications of the method. The viscous term that we need to define is then the turbulent term arrived at through Reynolds-averaging of the equations. If we let U represent the unsteady Reynolds-averaged velocity and u the turbulent fluctuations, the Reynolds stress terms that would be added to the right-hand-side of equations (16.a) and (16.b) are

$$-\frac{\partial}{\partial \xi^j}\overline{u^1 u^j} - \frac{2}{r_c}\overline{u^1\, u^2} + \frac{\epsilon\,\lambda}{r_c}\overline{u^1 u^1} \tag{1.a}$$

and

$$-\frac{\partial}{\partial \xi^j}\overline{u^2 u^j} + \frac{1}{r_c}\overline{u^1\, u^1} \tag{1.b}$$

and

$$-\frac{\partial}{\partial \xi^j}\overline{u^3 u^j} \tag{1.c}$$

One could make the modeling statements that

$$\sum_{j=1}^{3}\frac{\partial}{\partial \xi^j}\left(\nu_j \frac{\partial U^1}{\partial \xi^j}\right) = -\frac{\partial}{\partial \xi^j}\overline{u^1 u^j} - \frac{2}{r_c}\overline{u^1\, u^2} + \frac{\epsilon\,\lambda}{r_c}\overline{u^1 u^1} \tag{2.a}$$

$$\sum_{j=1}^{3}\frac{\partial}{\partial \xi^j}\left(\nu_j \frac{\partial U^2}{\partial \xi^j}\right) = -\frac{\partial}{\partial \xi^j}\overline{u^2 u^j} + \frac{1}{r_c}\overline{u^1\, u^1} \tag{2.b}$$

$$\sum_{j=1}^{3}\frac{\partial}{\partial \xi^j}\left(\nu_j \frac{\partial U^3}{\partial \xi^j}\right) = -\frac{\partial}{\partial \xi^j}\overline{u^3 u^j} \tag{2.c}$$

which would allow the standard treatment of turbulence as an eddy-viscosity. As often the horizontal eddy-viscosities in a river or estuary are treated a simple constants, the above approximation is likely to prove reasonable within the approximations of geophysical modeling. An approach that is arguably an improvement is to neglect the effect of the gradient of river curvature, in which case the model forms for the turbulence terms can be obtained from the cylindrical polar Navier-Stokes equations as:

$$\frac{\partial}{\partial r}\left(\frac{\nu_r}{r}\frac{\partial}{\partial r} r u_r\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\nu_\theta \frac{\partial u_r}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\nu_z \frac{\partial u_r}{\partial z}\right) - \frac{2\nu_\theta}{r^2}\frac{\partial u_\theta}{\partial \theta} \tag{3.a}$$

and

$$\frac{\partial}{\partial r}\left(\frac{\nu_r}{r}\frac{\partial}{\partial r}ru_\theta\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\nu_\theta\frac{\partial u_\theta}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(\nu_z\frac{\partial u_\theta}{\partial z}\right) + \frac{2\nu_\theta}{r^2}\frac{\partial u_r}{\partial\theta} \tag{3.b}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\nu_r r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\nu_\theta\frac{\partial u_z}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(\nu_z\frac{\partial u_z}{\partial z}\right) \tag{3.c}$$

The relation between the terms is

$$\begin{aligned}
r &= \delta + r_c && (4.a)\\
u_\theta &= \frac{r}{r_c}U^1 && (4.b)\\
u_r &= U^2 && (4.c)\\
u_z &= U^3 && (4.d)\\
\frac{\partial}{\partial\theta} &= r_c\frac{\partial}{\partial\xi^1} && (4.e)\\
\frac{\partial}{\partial r} &= \frac{\partial}{\partial\xi^2} && (4.f)\\
\frac{\partial}{\partial z} &= \frac{\partial}{\partial\xi^3} && (4.g)\\
\nu_r &= \nu_2 && (4.h)\\
\nu_\theta &= \nu_1 && (4.i)\\
\nu_z &= \nu_3 && (4.j)\\
& && (4.k)
\end{aligned}$$

Applying the relationship between the cylindrical-polar coordinate system and the curvilinear form we can write these as

$$\frac{\partial}{\partial\xi^2}\left(\frac{\nu_2}{r}\frac{\partial}{\partial\xi^2}rU^2\right) + \frac{r_c}{r^2}\frac{\partial}{\partial\xi^1}\left(r_c\nu_1\frac{\partial U^2}{\partial\xi^1}\right) + \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^2}{\partial z}\right) - \frac{2r_c\nu_1}{r^2}\frac{\partial}{\partial\xi^1}\left(\frac{r}{r_c}U^1\right) \tag{5.a}$$

and

$$\frac{\partial}{\partial\xi^2}\left(\frac{\nu_2}{r}\frac{\partial}{\partial\xi^2}\left[\frac{r^2}{r_c}U^1\right]\right) + \frac{r_c}{r^2}\frac{\partial}{\partial\xi^1}\left(r_c\nu_1\frac{\partial}{\partial\xi^1}\left[\frac{r}{r_c}U^1\right]\right) + \frac{\partial}{\partial z}\left(\nu_3\frac{\partial}{\partial z}\left[\frac{r}{r_c}U^1\right]\right) + \frac{2r_c\nu_1}{r^2}\frac{\partial U^2}{\partial\xi^1} \tag{5.b}$$

and

$$\frac{1}{r}\frac{\partial}{\partial\xi^2}\left(\nu_2 r\frac{\partial U^3}{\partial\xi^2}\right) + \frac{r_c}{r^2}\frac{\partial}{\partial\xi^1}\left(r_c\nu_1\frac{\partial U^3}{\partial\xi^1}\right) + \frac{\partial}{\partial\xi^3}\left(\nu_3\frac{\partial U^3}{\partial\xi^3}\right) \tag{5.c}$$

Expanding the derivatives we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^2}{\partial\xi^2} + \frac{\nu_2 U^2}{r}\frac{\partial r}{\partial\xi^2}\right) + \frac{r_c}{r^2}\left\{r_c\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^2}{\partial\xi^1}\right) + \nu_1\frac{\partial U^2}{\partial\xi^1}\frac{\partial r_c}{\partial\xi^1}\right\} + \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^2}{\partial z^2}\right)\\
&\qquad - \frac{2r_c\nu_1}{r^2}\left(\frac{r}{r_c}\right)\frac{\partial U^1}{\partial\xi^1} - \frac{2r_c\nu_1 U^1}{r^2}\frac{\partial}{\partial\xi^1}\left(\frac{r}{r_c}\right)
\end{aligned} \tag{6.a}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial\xi^2}\left(\frac{\nu_2}{r}\left[\frac{r^2}{r_c}\right]\frac{\partial U^1}{\partial\xi^2} + \frac{\nu_2 U^1}{r}\frac{\partial}{\partial\xi^2}\left[\frac{r^2}{r_c}\right]\right) + \frac{r_c}{r^2}\frac{\partial}{\partial\xi^1}\left(r_c\nu_1\left[\frac{r}{r_c}\right]\frac{\partial U^1}{\partial\xi^1} + r_c\nu_1 U^1\frac{\partial}{\partial\xi^1}\left[\frac{r}{r_c}\right]\right)\\
&\qquad + \frac{r}{r_c}\frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^1}{\partial z}\right) + \frac{2r_c\nu_1}{r^2}\frac{\partial U^2}{\partial\xi^1}
\end{aligned} \tag{6.b}$$

and

$$\frac{1}{r}\left\{\frac{\partial r}{\partial\xi^2}\left(\nu_2\frac{\partial U^3}{\partial\xi^2}\right) + r\frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^3}{\partial\xi^2}\right)\right\} + \frac{r_c}{r^2}\left\{\frac{\partial r_c}{\partial\xi^1}\left(\nu_1\frac{\partial U^3}{\partial\xi^1}\right) + r_c\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^3}{\partial\xi^1}\right)\right\} + \frac{\partial}{\partial\xi^3}\left(\nu_3\frac{\partial U^3}{\partial\xi^3}\right) \tag{6.c}$$

Cancellations give

$$\frac{\partial}{\partial \xi^2}\left(\nu_2 \frac{\partial U^2}{\partial \xi^2} + \frac{\nu_2 U^2}{r}\frac{\partial r}{\partial \xi^2}\right) + \frac{r_c^2}{r^2}\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^2}{\partial \xi^1}\right) + \frac{\nu_1\, r_c}{r^2}\frac{\partial U^2}{\partial \xi^1}\frac{\partial r_c}{\partial \xi^1} + \frac{\partial}{\partial z}\left(\nu_3 \frac{\partial U^2}{\partial z}\right)$$
$$- \frac{2\nu_1}{r}\frac{\partial U^1}{\partial \xi^1} - \frac{2 r_c \nu_1\, U^1}{r^2}\frac{\partial}{\partial \xi^1}\left(\frac{r}{r_c}\right) \tag{7.a}$$

and

$$\frac{\partial}{\partial \xi^2}\left(\frac{r\nu_2}{r_c}\frac{\partial U^1}{\partial \xi^2} + \frac{\nu_2 U^1}{r}\frac{\partial}{\partial \xi^2}\left[\frac{r^2}{r_c}\right]\right) + \frac{r_c}{r^2}\frac{\partial}{\partial \xi^1}\left(r\nu_1 \frac{\partial U^1}{\partial \xi^1} + r_c \nu_1 U^1 \frac{\partial}{\partial \xi^1}\left[\frac{r}{r_c}\right]\right)$$
$$+ \frac{r}{r_c}\frac{\partial}{\partial z}\left(\nu_3 \frac{\partial U^1}{\partial z}\right) + \frac{2 r_c \nu_1}{r^2}\frac{\partial U^2}{\partial \xi^1} \tag{7.b}$$

and

$$\frac{1}{r}\frac{\partial r}{\partial \xi^2}\left(\nu_2 \frac{\partial U^3}{\partial \xi^2}\right) + \frac{\partial}{\partial \xi^2}\left(\nu_2 \frac{\partial U^3}{\partial \xi^2}\right) + \frac{r_c}{r^2}\frac{\partial r_c}{\partial \xi^1}\left(\nu_1 \frac{\partial U^3}{\partial \xi^1}\right) + \frac{r_c^2}{r^2}\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^3}{\partial \xi^1}\right) + \frac{\partial}{\partial \xi^3}\left(\nu_3 \frac{\partial U^3}{\partial \xi^3}\right) \tag{7.c}$$

Note that

$$\frac{\partial r_c}{\partial \xi^1} = \lambda \tag{8.a}$$
$$\frac{\partial r}{\partial \xi^1} = \frac{\partial \delta}{\partial \xi^1} + \frac{\partial r_c}{\partial \xi^1} = \lambda \tag{8.b}$$
$$\frac{\partial \epsilon}{\partial \xi^1} = \frac{\partial}{\partial \xi^1}\left(\frac{r}{r_c}\right) \tag{8.c}$$
$$\frac{\partial}{\partial \xi^1}\left(\frac{r}{r_c}\right) = \frac{1}{r_c}\frac{\partial r}{\partial \xi^1} + r\frac{\partial r_c^{-1}}{\partial \xi^1} = \frac{\lambda}{r_c} - \frac{r}{r_c^2}\frac{\partial r_c}{\partial \xi^1} = \frac{\lambda}{r_c} - \frac{r\lambda}{r_c^2} = \lambda\left(\frac{r_c - r}{r_c^2}\right)$$
$$= -\frac{\lambda\,\epsilon}{r_c} \tag{8.d}$$
$$\frac{\partial r}{\partial \xi^2} = 1 \tag{8.e}$$
$$\frac{\partial r_c}{\partial \xi^2} = 0 \tag{8.f}$$
$$\frac{\partial r^{-1}}{\partial \xi^2} = -\frac{1}{r^2}\frac{\partial r}{\partial \xi^2} = -\frac{1}{r^2} \tag{8.g}$$
$$\frac{\partial r_c^{-1}}{\partial \xi^2} = -\frac{1}{r_c^2}\frac{\partial r_c}{\partial \xi^2} = 0 \tag{8.h}$$
$$\frac{\partial}{\partial \xi^2}\left(\frac{r}{r_c}\right) = \frac{1}{r_c}\frac{\partial r}{\partial \xi^2} + r\frac{\partial r_c^{-1}}{\partial \xi^2} = \frac{1}{r_c} - \frac{r}{r_c^2}\frac{\partial r_c}{\partial \xi^2} = \frac{1}{r_c} \tag{8.i}$$
$$\frac{\partial}{\partial \xi^2}\left(\frac{r^2}{r_c}\right) = \frac{1}{r_c}\frac{\partial r^2}{\partial \xi^2} = \frac{2r}{r_c}\frac{\partial r}{\partial \xi^2} = \frac{2r}{r_c} \tag{8.j}$$

So further simplification leads to

$$\frac{\partial}{\partial \xi^2}\left(\nu_2 \frac{\partial U^2}{\partial \xi^2} + \frac{\nu_2 U^2}{r}\right) + \frac{r_c^2}{r^2}\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^2}{\partial \xi^1}\right) + \frac{\nu_1\,\lambda\, r_c}{r^2}\frac{\partial U^2}{\partial \xi^1}$$
$$+ \frac{\partial}{\partial z}\left(\nu_3 \frac{\partial U^2}{\partial z}\right) - \frac{2\nu_1}{r}\frac{\partial U^1}{\partial \xi^1} + \frac{2\lambda\,\epsilon\,\nu_1\, U^1}{r^2} \tag{9.a}$$

and

$$\frac{\partial}{\partial \xi^2}\left(\frac{r\nu_2}{r_c}\frac{\partial U^1}{\partial \xi^2} + \frac{2\nu_2 U^1}{r_c}\right) + \frac{r_c}{r^2}\frac{\partial}{\partial \xi^1}\left(r\nu_1 \frac{\partial U^1}{\partial \xi^1} - \lambda\,\epsilon\,\nu_1 U^1\right)$$
$$+ \frac{r}{r_c}\frac{\partial}{\partial z}\left(\nu_3 \frac{\partial U^1}{\partial z}\right) + \frac{2 r_c \nu_1}{r^2}\frac{\partial U^2}{\partial \xi^1} \tag{9.b}$$

and

$$\frac{1}{r}\left(\nu_2\frac{\partial U^3}{\partial \xi^2}\right) + \frac{\partial}{\partial \xi^2}\left(\nu_2\frac{\partial U^3}{\partial \xi^2}\right) + \frac{r_c}{r^2}\lambda\left(\nu_1\frac{\partial U^3}{\partial \xi^1}\right) + \frac{r_c^2}{r^2}\frac{\partial}{\partial \xi^1}\left(\nu_1\frac{\partial U^3}{\partial \xi^1}\right) + \frac{\partial}{\partial \xi^3}\left(\nu_3\frac{\partial U^3}{\partial \xi^3}\right) \quad (9.c)$$

Cancellation and distribution gives

$$\frac{\partial}{\partial \xi^2}\left(\nu_2\frac{\partial U^2}{\partial \xi^2}\right) + \frac{\partial}{\partial \xi^2}\left(\frac{\nu_2 U^2}{r}\right) + \frac{r_c^2}{r^2}\frac{\partial}{\partial \xi^1}\left(\nu_1\frac{\partial U^2}{\partial \xi^1}\right) + \frac{\nu_1\,\lambda\,r_c}{r^2}\frac{\partial U^2}{\partial \xi^1}$$
$$+ \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^2}{\partial z}\right) - \frac{2\nu_1}{r}\frac{\partial U^1}{\partial \xi^1} + \frac{2\lambda\,\epsilon\,\nu_1\,U^1}{r^2} \quad (10.a)$$

and

$$\frac{\partial}{\partial \xi^2}\left(\frac{r\nu_2}{r_c}\frac{\partial U^1}{\partial \xi^2}\right) + \frac{\partial}{\partial \xi^2}\left(\frac{2\nu_2 U^1}{r_c}\right) + \frac{r_c}{r^2}\frac{\partial}{\partial \xi^1}\left(r\nu_1\frac{\partial U^1}{\partial \xi^1}\right) - \frac{r_c}{r^2}\frac{\partial}{\partial \xi^1}\left(\lambda\,\epsilon\,\nu_1 U^1\right)$$
$$+ \frac{r}{r_c}\frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^1}{\partial z}\right) + \frac{2r_c\nu_1}{r^2}\frac{\partial U^2}{\partial \xi^1} \quad (10.b)$$

with no further changes to equation (9.c) for the moment.

Some further manipulations

$$\frac{\partial}{\partial \xi^2}\left(\nu_2\frac{\partial U^2}{\partial \xi^2}\right) + \frac{1}{r}\frac{\partial}{\partial \xi^2}\left(\nu_2 U^2\right) + \nu_2 U^2\frac{\partial r^{-1}}{\partial \xi^2} + \frac{r_c^2}{r^2}\frac{\partial}{\partial \xi^1}\left(\nu_1\frac{\partial U^2}{\partial \xi^1}\right) + \frac{\nu_1\,\lambda\,r_c}{r^2}\frac{\partial U^2}{\partial \xi^1}$$
$$+ \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^2}{\partial z}\right) - \frac{2\nu_1}{r}\frac{\partial U^1}{\partial \xi^1} + \frac{2\lambda\,\epsilon\,\nu_1\,U^1}{r^2} \quad (11.a)$$

and

$$\frac{r}{r_c}\frac{\partial}{\partial \xi^2}\left(\nu_2\frac{\partial U^1}{\partial \xi^2}\right) + \nu_2\frac{\partial U^1}{\partial \xi^2}\frac{\partial}{\partial \xi^2}\left(\frac{r}{r_c}\right) + \frac{2}{r_c}\frac{\partial}{\partial \xi^2}\left(\nu_2 U^1\right) + 2\nu_2 U^1\frac{\partial r_c^{-1}}{\partial \xi^2}$$
$$+ \frac{r_c}{r}\frac{\partial}{\partial \xi^1}\left(\nu_1\frac{\partial U^1}{\partial \xi^1}\right) + \nu_1\frac{\partial U^1}{\partial \xi^1}\frac{r_c}{r^2}\frac{\partial r}{\partial \xi^1} - \lambda\,\epsilon\frac{r_c}{r^2}\frac{\partial}{\partial \xi^1}\left(\nu_1 U^1\right) - \nu_1 U^1\frac{r_c}{r^2}\left(\epsilon\frac{\partial \lambda}{\partial \xi^1} + \lambda\frac{\partial \epsilon}{\partial \xi^1}\right)$$
$$+ \frac{r}{r_c}\frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^1}{\partial z}\right) + \frac{2r_c\nu_1}{r^2}\frac{\partial U^2}{\partial \xi^1} \quad (11.b)$$

It follows that

$$\frac{\partial}{\partial \xi^2}\left(\nu_2\frac{\partial U^2}{\partial \xi^2}\right) + \frac{1}{r}\frac{\partial}{\partial \xi^2}\left(\nu_2 U^2\right) - \frac{\nu_2 U^2}{r^2} + \frac{r_c^2}{r^2}\frac{\partial}{\partial \xi^1}\left(\nu_1\frac{\partial U^2}{\partial \xi^1}\right) + \frac{\nu_1\,\lambda\,r_c}{r^2}\frac{\partial U^2}{\partial \xi^1}$$
$$+ \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^2}{\partial z}\right) - \frac{2\nu_1}{r}\frac{\partial U^1}{\partial \xi^1} + \frac{2\lambda\,\epsilon\,\nu_1\,U^1}{r^2} \quad (12.a)$$

and

$$\frac{r}{r_c}\frac{\partial}{\partial \xi^2}\left(\nu_2\frac{\partial U^1}{\partial \xi^2}\right) + \frac{\nu_2}{r_c}\frac{\partial U^1}{\partial \xi^2} + \frac{2}{r_c}\frac{\partial}{\partial \xi^2}\left(\nu_2 U^1\right)$$
$$+ \frac{r_c}{r}\frac{\partial}{\partial \xi^1}\left(\nu_1\frac{\partial U^1}{\partial \xi^1}\right) + \frac{\nu_1\,r_c\,\lambda}{r^2}\frac{\partial U^1}{\partial \xi^1} - \lambda\,\epsilon\frac{r_c}{r^2}\frac{\partial}{\partial \xi^1}\left(\nu_1 U^1\right) - \nu_1 U^1\frac{r_c}{r^2}\left(\epsilon\frac{\partial \lambda}{\partial \xi^1} - \frac{\epsilon\lambda^2}{r_c}\right)$$
$$+ \frac{r}{r_c}\frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^1}{\partial z}\right) + \frac{2r_c\nu_1}{r^2}\frac{\partial U^2}{\partial \xi^1} \quad (12.b)$$

Some rearranging gives

$$\frac{r_c^2}{r^2}\frac{\partial}{\partial \xi^1}\left(\nu_1\frac{\partial U^2}{\partial \xi^1}\right) + \frac{\partial}{\partial \xi^2}\left(\nu_2\frac{\partial U^2}{\partial \xi^2}\right) + \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^2}{\partial z}\right)$$
$$U^1\left(\frac{2\lambda\,\epsilon\,\nu_1}{r^2}\right) + U^2\left(\frac{1}{r}\frac{\partial \nu_2}{\partial \xi^2} - \frac{\nu_2}{r^2}\right)$$
$$- \frac{2\nu_1}{r}\frac{\partial U^1}{\partial \xi^1} + \frac{\nu_2}{r}\frac{\partial U^2}{\partial \xi^2} + \frac{\nu_1\,\lambda\,r_c}{r^2}\frac{\partial U^2}{\partial \xi^1} \quad (13.a)$$

and

$$\frac{r_c}{r}\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^1}{\partial\xi^1}\right) + \frac{r}{r_c}\frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^1}{\partial\xi^2}\right) + \frac{r}{r_c}\frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^1}{\partial z}\right)$$
$$+ U^1\left(\frac{2}{r_c}\frac{\partial\nu_2}{\partial\xi^2} - \lambda\,\epsilon\,\frac{r_c}{r^2}\frac{\partial\nu_1}{\partial\xi^1} - \nu_1\frac{r_c}{r^2}\left[\epsilon\frac{\partial\lambda}{\partial\xi^1} - \frac{\epsilon\lambda^2}{r_c}\right]\right)$$
$$\frac{\partial U^1}{\partial\xi^1}\left(\frac{\nu_1\,r_c\,\lambda}{r^2} - \nu_1\,\lambda\,\epsilon\,\frac{r_c}{r^2}\right) + \frac{\partial U^1}{\partial\xi^2}\left(\frac{\nu_2}{r_c} + \frac{2\nu_2}{r_c}\right) + \frac{2r_c\nu_1}{r^2}\frac{\partial U^2}{\partial\xi^1} \tag{13.b}$$

Noting that $r = r_c\,(1+\epsilon)$ we have

$$\frac{1}{(1+\epsilon)^2}\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^2}{\partial\xi^1}\right) + \frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^2}{\partial\xi^2}\right) + \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^2}{\partial z}\right)$$
$$U^1\left(\frac{2\lambda\,\epsilon\,\nu_1}{r_c^2\,[1+\epsilon]^2}\right) + U^2\left(\frac{1}{r_c\,[1+\epsilon]}\frac{\partial\nu_2}{\partial\xi^2} - \frac{\nu_2}{r_c^2\,[1+\epsilon]^2}\right)$$
$$- \frac{2\nu_1}{r_c\,(1+\epsilon)}\frac{\partial U^1}{\partial\xi^1} + \frac{\nu_2}{r_c\,(1+\epsilon)}\frac{\partial U^2}{\partial\xi^2} + \frac{\nu_1\,\lambda}{r_c\,(1+\epsilon)^2}\frac{\partial U^2}{\partial\xi^1} \tag{14.a}$$

and

$$\frac{1}{1+\epsilon}\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^1}{\partial\xi^1}\right) + (1+\epsilon)\frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^1}{\partial\xi^2}\right) + (1+\epsilon)\frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^1}{\partial z}\right)$$
$$+ U^1\left(\frac{2}{r_c}\frac{\partial\nu_2}{\partial\xi^2} - \frac{\lambda\,\epsilon}{r_c\,[1+\epsilon]^2}\frac{\partial\nu_1}{\partial\xi^1} - \nu_1\frac{1}{r_c\,[1+\epsilon]^2}\left[\epsilon\frac{\partial\lambda}{\partial\xi^1} - \frac{\epsilon\lambda^2}{r_c}\right]\right)$$
$$\frac{\partial U^1}{\partial\xi^1}\left(\frac{\nu_1\,\lambda}{r_c\,[1+\epsilon]^2} - \frac{\nu_1\,\lambda\,\epsilon}{r_c\,(1+\epsilon)^2}\right) + \frac{\partial U^1}{\partial\xi^2}\left(\frac{3\nu_2}{r_c}\right) + \frac{2\nu_1}{r_c\,(1+\epsilon)^2}\frac{\partial U^2}{\partial\xi^1} \tag{14.b}$$

with 9.c becoming

$$\frac{r_c^2}{r_c^2\,(1+\epsilon)^2}\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^3}{\partial\xi^1}\right) + \frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^3}{\partial\xi^2}\right) + \frac{\partial}{\partial\xi^3}\left(\nu_3\frac{\partial U^3}{\partial\xi^3}\right)$$
$$\frac{1}{r_c\,(1+\epsilon)}\left(\nu_2\frac{\partial U^3}{\partial\xi^2}\right) + \frac{r_c}{r_c^2\,(1+\epsilon)^2}\lambda\left(\nu_1\frac{\partial U^3}{\partial\xi^1}\right) \tag{14.c}$$

Using the binomial expansion relations for small ϵ we obtain

$$(1-2\epsilon)\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^2}{\partial\xi^1}\right) + \frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^2}{\partial\xi^2}\right) + \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^2}{\partial z}\right)$$
$$U^1\left([1-2\epsilon]\frac{2\lambda\,\epsilon\,\nu_1}{r_c^2}\right) + U^2\left(\frac{[1-\epsilon]}{r_c}\frac{\partial\nu_2}{\partial\xi^2} - \frac{\nu_2\,[1-2\epsilon]}{r_c^2}\right)$$
$$- [1-\epsilon]\frac{2\nu_1}{r_c}\frac{\partial U^1}{\partial\xi^1} + [1-\epsilon]\frac{\nu_2}{r_c}\frac{\partial U^2}{\partial\xi^2} + [1-2\epsilon]\frac{\nu_1\,\lambda}{r_c}\frac{\partial U^2}{\partial\xi^1} + O\left(\epsilon^2\right) \tag{15.a}$$

and

$$(1-\epsilon)\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^1}{\partial\xi^1}\right) + (1+\epsilon)\frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^1}{\partial\xi^2}\right) + (1+\epsilon)\frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^1}{\partial z}\right)$$
$$+ U^1\left(\frac{2}{r_c}\frac{\partial\nu_2}{\partial\xi^2} - [1-2\epsilon]\frac{\lambda\,\epsilon}{r_c}\frac{\partial\nu_1}{\partial\xi^1} - \nu_1\frac{[1-2\epsilon]}{r_c}\left[\epsilon\frac{\partial\lambda}{\partial\xi^1} - \frac{\epsilon\lambda^2}{r_c}\right]\right)$$
$$[1-2\epsilon]\frac{\partial U^1}{\partial\xi^1}\left(\frac{\nu_1\,\lambda\,[1-\epsilon]}{r_c}\right) + \frac{\partial U^1}{\partial\xi^2}\left(\frac{3\nu_2}{r_c}\right) + (1-2\epsilon)\frac{2\nu_1}{r_c}\frac{\partial U^2}{\partial\xi^1} + O\left(\epsilon^2\right) \tag{15.b}$$

and

$$(1-2\epsilon)\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^3}{\partial\xi^1}\right) \;+\; \frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^3}{\partial\xi^2}\right) + \frac{\partial}{\partial\xi^3}\left(\nu_3\frac{\partial U^3}{\partial\xi^3}\right)$$
$$\left(\frac{1}{r_c}+\frac{\epsilon}{r_c}\right)\left(\nu_2\frac{\partial U^3}{\partial\xi^2}\right) + \frac{\lambda}{r_c}(1-2\epsilon)\left(\nu_1\frac{\partial U^3}{\partial\xi^1}\right) + O\left(\epsilon^2\right) \quad (15.c)$$

If we neglect terms of order ϵ^2, r_c^{-2}, ϵr_c^{-1} and $\epsilon r_c^{-2}\lambda$ and $\epsilon^2 r_c^{-1}\lambda$ and higher (under the presumption that $\lambda \sim O\left(\epsilon^{-1}\right)$), we obtain

$$(1-2\epsilon)\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^2}{\partial\xi^1}\right) + \frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^2}{\partial\xi^2}\right) + \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^2}{\partial z}\right)$$
$$+U^2\left(\frac{1}{r_c}\frac{\partial\nu_2}{\partial\xi^2}\right) - \frac{2\nu_1}{r_c}\frac{\partial U^1}{\partial\xi^1} + \frac{\nu_2}{r_c}\frac{\partial U^2}{\partial\xi^2} + [1-2\epsilon]\frac{\nu_1\lambda}{r_c}\frac{\partial U^2}{\partial\xi^1} + O\left(\epsilon^2\right) \quad (16.a)$$

and

$$(1-\epsilon)\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^1}{\partial\xi^1}\right) + (1+\epsilon)\frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^1}{\partial\xi^2}\right) + (1+\epsilon)\frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^1}{\partial z}\right)$$
$$+U^1\left(\frac{2}{r_c}\frac{\partial\nu_2}{\partial\xi^2} - \frac{\lambda\,\epsilon}{r_c}\frac{\partial\nu_1}{\partial\xi^1} - \frac{\nu_1}{r_c}\left[\epsilon\frac{\partial\lambda}{\partial\xi^1} - \frac{\epsilon\lambda^2}{r_c}\right]\right)$$
$$[1-3\epsilon]\frac{\partial U^1}{\partial\xi^1}\left(\frac{\nu_1\lambda}{r_c}\right) + \frac{\partial U^1}{\partial\xi^2}\left(\frac{3\nu_2}{r_c}\right) + \frac{2\nu_1}{r_c}\frac{\partial U^2}{\partial\xi^1} + O\left(\epsilon^2\right) \quad (16.b)$$

and

$$(1-2\epsilon)\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^3}{\partial\xi^1}\right) \;+\; \frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^3}{\partial\xi^2}\right) + \frac{\partial}{\partial\xi^3}\left(\nu_3\frac{\partial U^3}{\partial\xi^3}\right)$$
$$\frac{1}{r_c}\left(\nu_2\frac{\partial U^3}{\partial\xi^2}\right) + \frac{\lambda}{r_c}(1-2\epsilon)\left(\nu_1\frac{\partial U^3}{\partial\xi^1}\right) + O\left(\epsilon^2\right) \quad (16.c)$$

Some regrouping of terms

$$(1-2\epsilon)\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^2}{\partial\xi^1}\right) + \frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^2}{\partial\xi^2}\right) + \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^2}{\partial z}\right)$$
$$+\frac{1}{r_c}\frac{\partial}{\partial\xi^2}\left(\nu_2 U^2\right) - \frac{2\nu_1}{r_c}\frac{\partial U^1}{\partial\xi^1} + [1-2\epsilon]\frac{\nu_1\lambda}{r_c}\frac{\partial U^2}{\partial\xi^1} + O\left(\epsilon^2\right) \quad (17.a)$$

and

$$(1-\epsilon)\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^1}{\partial\xi^1}\right) + (1+\epsilon)\frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^1}{\partial\xi^2}\right) + (1+\epsilon)\frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^1}{\partial z}\right)$$
$$+\frac{3}{r_c}\frac{\partial}{\partial\xi^2}\left(\nu_2 U^1\right) - U^1\left(\frac{1}{r_c}\frac{\partial\nu_2}{\partial\xi^2} + \frac{\lambda\,\epsilon}{r_c}\frac{\partial\nu_1}{\partial\xi^1} + \frac{\nu_1}{r_c}\left[\epsilon\frac{\partial\lambda}{\partial\xi^1} - \frac{\epsilon\lambda^2}{r_c}\right]\right)$$
$$[1-3\epsilon]\frac{\partial U^1}{\partial\xi^1}\left(\frac{\nu_1\lambda}{r_c}\right) + \frac{2\nu_1}{r_c}\frac{\partial U^2}{\partial\xi^1} + O\left(\epsilon^2\right) \quad (17.b)$$

We began with the cylindrical polar coordinate form, but derived terms including λ which should be exactly zero for the cylindrical polar form. These terms arise in the transformation. However, it is possible that these terms would be offset by other λ terms that would be obtained from a derivation from the curvilinear form. Thus, it is reasonable for the purposes of developing a turbulence model to neglect terms with λ and obtain

$$(1-2\epsilon)\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^2}{\partial\xi^1}\right) + \frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^2}{\partial\xi^2}\right) + \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^2}{\partial z}\right)$$
$$+\frac{1}{r_c}\frac{\partial}{\partial\xi^2}\left(\nu_2 U^2\right) - \frac{2\nu_1}{r_c}\frac{\partial U^1}{\partial\xi^1} + O\left(\epsilon^2\right) \quad (18.a)$$

and

$$(1-\epsilon)\,\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^1}{\partial \xi^1}\right) \;+\; (1+\epsilon)\,\frac{\partial}{\partial \xi^2}\left(\nu_2 \frac{\partial U^1}{\partial \xi^2}\right) \;+\; (1+\epsilon)\,\frac{\partial}{\partial z}\left(\nu_3 \frac{\partial U^1}{\partial z}\right)$$
$$+\;\frac{3}{r_c}\frac{\partial}{\partial \xi^2}\Big(\nu_2 U^1\Big) \;-\; U^1\left(\frac{1}{r_c}\frac{\partial \nu_2}{\partial \xi^2}\right) \;+\; \frac{2\nu_1}{r_c}\frac{\partial U^2}{\partial \xi^1} \;+\; O\left(\epsilon^2\right) \tag{18.b}$$

and

$$(1-2\epsilon)\,\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^3}{\partial \xi^1}\right) \;+\; \frac{\partial}{\partial \xi^2}\left(\nu_2 \frac{\partial U^3}{\partial \xi^2}\right) \;+\; \frac{\partial}{\partial \xi^3}\left(\nu_3 \frac{\partial U^3}{\partial \xi^3}\right)$$
$$\frac{1}{r_c}\left(\nu_2 \frac{\partial U^3}{\partial \xi^2}\right) \;+\; O\left(\epsilon^2\right) \tag{18.c}$$

Some further modifications provides

$$\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^2}{\partial \xi^1}\right) \;+\; \frac{\partial}{\partial \xi^2}\left(\nu_2 \frac{\partial U^2}{\partial \xi^2}\right) \;+\; \frac{\partial}{\partial z}\left(\nu_3 \frac{\partial U^2}{\partial z}\right)$$
$$-\;\frac{2}{r_c}\frac{\partial}{\partial \xi^1}\Big(\nu_1 U^1\Big) \;+\; \frac{1}{r_c}\frac{\partial}{\partial \xi^2}\Big(\nu_2 U^2\Big) \;+\; \frac{2U^1}{r_c}\frac{\partial \nu_1}{\partial \xi^1} \;-\; 2\epsilon\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^2}{\partial \xi^1}\right) \;+\; O\left(\epsilon^2\right) \tag{19.a}$$

and

$$\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^1}{\partial \xi^1}\right) \;+\; \frac{\partial}{\partial \xi^2}\left(\nu_2 \frac{\partial U^1}{\partial \xi^2}\right) \;+\; \frac{\partial}{\partial z}\left(\nu_3 \frac{\partial U^1}{\partial z}\right)$$
$$-\,\epsilon\left\{\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^1}{\partial \xi^1}\right) \;-\; \frac{\partial}{\partial \xi^2}\left(\nu_2 \frac{\partial U^1}{\partial \xi^2}\right) \;-\; \frac{\partial}{\partial z}\left(\nu_3 \frac{\partial U^1}{\partial z}\right)\right\}$$
$$+\;\frac{3}{r_c}\frac{\partial}{\partial \xi^2}\Big(\nu_2 U^1\Big) \;+\; \frac{2}{r_c}\frac{\partial}{\partial \xi^1}\Big(\nu_1 U^2\Big) \;-\; U^1\left(\frac{1}{r_c}\frac{\partial \nu_2}{\partial \xi^2}\right) \;-\; \frac{2U^2}{r_c}\frac{\partial \nu_1}{\partial \xi^1} \;+\; O\left(\epsilon^2\right) \tag{19.b}$$

and

$$\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^3}{\partial \xi^1}\right) \;+\; \frac{\partial}{\partial \xi^2}\left(\nu_2 \frac{\partial U^3}{\partial \xi^2}\right) \;+\; \frac{\partial}{\partial \xi^3}\left(\nu_3 \frac{\partial U^3}{\partial \xi^3}\right)$$
$$+\;\frac{1}{r_c}\left(\nu_2 \frac{\partial U^3}{\partial \xi^2}\right) \;-\; 2\epsilon\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^3}{\partial \xi^1}\right) \;+\; O\left(\epsilon^2\right) \tag{19.c}$$

Under the conditions that $\partial \nu_1/\partial \xi^j \sim O(\epsilon)$ and $\partial \nu_2/\partial \xi^j \sim O(\epsilon)$ we obtain

$$\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^2}{\partial \xi^1}\right) \;+\; \frac{\partial}{\partial \xi^2}\left(\nu_2 \frac{\partial U^2}{\partial \xi^2}\right) \;+\; \frac{\partial}{\partial z}\left(\nu_3 \frac{\partial U^2}{\partial z}\right)$$
$$-\;\frac{2\nu_1}{r_c}\frac{\partial U^1}{\partial \xi^1} \;+\; \frac{\nu_2}{r_c}\frac{\partial U^2}{\partial \xi^2} \;-\; 2\nu_1\,\epsilon\frac{\partial^2 U^2}{\partial \xi^1 \partial \xi^1} \;+\; O\left(\epsilon^2\right) \tag{20.a}$$

and

$$\frac{\partial}{\partial \xi^1}\left(\nu_1 \frac{\partial U^1}{\partial \xi^1}\right) \;+\; \frac{\partial}{\partial \xi^2}\left(\nu_2 \frac{\partial U^1}{\partial \xi^2}\right) \;+\; \frac{\partial}{\partial z}\left(\nu_3 \frac{\partial U^1}{\partial z}\right)$$
$$-\,\epsilon\left\{\nu_1\frac{\partial^2 U^1}{\partial \xi^1 \partial \xi^1} \;-\; \nu_2\frac{\partial U^1}{\partial \xi^2 \partial \xi^2} \;-\; \frac{\partial}{\partial z}\left(\nu_3 \frac{\partial U^1}{\partial z}\right)\right\}$$
$$+\;\frac{3\nu_2}{r_c}\frac{\partial U^1}{\partial \xi^2} \;+\; \frac{2\nu_1}{r_c}\frac{\partial U^2}{\partial \xi^1} \;+\; O\left(\epsilon^2\right) \tag{20.b}$$

and

$$\frac{\partial}{\partial\xi^1}\left(\nu_1\frac{\partial U^3}{\partial\xi^1}\right) + \frac{\partial}{\partial\xi^2}\left(\nu_2\frac{\partial U^3}{\partial\xi^2}\right) + \frac{\partial}{\partial\xi^3}\left(\nu_3\frac{\partial U^3}{\partial\xi^3}\right) + \frac{\nu_2}{r_c}\frac{\partial U^3}{\partial\xi^2} - 2\epsilon\nu_1\frac{\partial^2 U^3}{\partial\xi^1\xi^1} + O\left(\epsilon^2\right) \tag{20.c}$$

So we finally obtain the eddy-viscosity turbulence model formulation as

$$-\frac{\partial}{\partial\xi^j}\overline{u^2u^j} + \frac{1}{r_c}\overline{u^1\,u^1} \approx \sum_{j=1}^{3}\frac{\partial}{\partial\xi^j}\left(\nu_j\frac{\partial U^2}{\partial\xi^j}\right) - \frac{2\nu_1}{r_c}\frac{\partial U^1}{\partial\xi^1} + \frac{\nu_2}{r_c}\frac{\partial U^2}{\partial\xi^2} - 2\nu_1\,\epsilon\frac{\partial^2 U^2}{\partial\xi^1\partial\xi^1} + O\left(\epsilon^2\right) \tag{21.a}$$

and

$$-\frac{\partial}{\partial\xi^j}\overline{u^1u^j} - \frac{2}{r_c}\overline{u^1\,u^2} + \frac{\epsilon\,\lambda}{r_c}\overline{u^1u^1} \approx \sum_{j=1}^{3}\frac{\partial}{\partial\xi^j}\left(\nu_j\frac{\partial U^1}{\partial\xi^j}\right) - \epsilon\left\{\nu_1\frac{\partial^2 U^1}{\partial\xi^1\partial\xi^1} - \nu_2\frac{\partial U^1}{\partial\xi^2\partial\xi^2} - \frac{\partial}{\partial z}\left(\nu_3\frac{\partial U^1}{\partial z}\right)\right\} + \frac{3\nu_2}{r_c}\frac{\partial U^1}{\partial\xi^2} + \frac{2\nu_1}{r_c}\frac{\partial U^2}{\partial\xi^1} + O\left(\epsilon^2\right) \tag{21.b}$$

and

$$-\frac{\partial}{\partial\xi^j}\overline{u^3u^j} \approx \sum_{j=1}^{3}\frac{\partial}{\partial\xi^j}\left(\nu_j\frac{\partial U^3}{\partial\xi^j}\right) + \frac{\nu_2}{r_c}\frac{\partial U^3}{\partial\xi^2} - 2\epsilon\nu_1\frac{\partial^2 U^3}{\partial\xi^1\xi^1} + O\left(\epsilon^2\right) \tag{21.c}$$

Chapter 5

Continuity and the kinematic boundary condition

5.1 Curvilinear continuity in the grid-stretched form

The following approach applies the grid stretching and perturbation expansion to the continuity equation. However, this was found to provide poor results. A better approach is found in Hodges and Imberger (2000).

The continuity equation is (aris pg 178 eq 8.12.3)

$$U^i_{,i} \;=\; 0 \tag{1}$$

$$U^i_{,i} \;=\; \frac{\partial U^1}{\partial \xi^1} + \frac{\partial U^2}{\partial \xi^2} + \frac{\partial U^3}{\partial \xi^3} + \frac{1+\gamma_2}{2J^2}\left[U^1\frac{\partial \gamma_1}{\partial \xi^1} + U^2\frac{\partial \gamma_1}{\partial \xi^2}\right] + \frac{1+\gamma_1}{2J^2}\left[U^1\frac{\partial \gamma_2}{\partial \xi^1} + U^2\frac{\partial \gamma_2}{\partial \xi^2}\right] \tag{2}$$

5.2 Reduction to cylindrical polar coordinates

first remove γ_2 terms and ξ^1 gradient of γ_1

$$\frac{1}{r_c}\frac{\partial}{\partial \theta}\left(\frac{r_c}{r}u_\theta\right) + \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{1}{2J^2}u_r\frac{\partial \gamma_1}{\partial \xi^2} \;=\; 0 \tag{3}$$

Next substitute other relations

$$\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{r_c^2}{2r^2}u_r\frac{2r}{r_c^2} \;=\; 0 \tag{4}$$

Arrive at

$$\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} \;=\; 0 \tag{5}$$

5.3 The curvilinear perturbation form of continuity

$$\frac{\partial U^1}{\partial \xi^1} + \frac{\partial U^2}{\partial \xi^2} + \frac{\partial U^3}{\partial \xi^3} + \frac{r_c^2}{2r^2}\left[-U^1\frac{2\epsilon\lambda}{r_c}(1+\epsilon) + U^2\frac{2}{r_c}(1+\epsilon)\right] \;=\; 0 \tag{6}$$

or noting that $1+\epsilon = r/r_c$

$$\frac{\partial U^1}{\partial \xi^1} + \frac{\partial U^2}{\partial \xi^2} + \frac{\partial U^3}{\partial \xi^3} + \frac{1}{2\left(1+\epsilon\right)^2}\left[- U^1 \frac{2\epsilon\lambda}{r_c}\left(1+\epsilon\right) + U^2 \frac{2}{r_c}\left(1+\epsilon\right)\right] = 0 \tag{7}$$

or

$$\frac{\partial U^1}{\partial \xi^1} + \frac{\partial U^2}{\partial \xi^2} + \frac{\partial U^3}{\partial \xi^3} + \frac{1}{(1+\epsilon)}\left[- U^1 \frac{\epsilon\lambda}{r_c} + U^2 \frac{1}{r_c}\right] = 0 \tag{8}$$

with binomial expansion

$$\frac{\partial U^1}{\partial \xi^1} + \frac{\partial U^2}{\partial \xi^2} + \frac{\partial U^3}{\partial \xi^3} + (1-\epsilon)\left[- U^1 \frac{\epsilon\lambda}{r_c} + U^2 \frac{1}{r_c}\right] + O\left(\epsilon^2\right) = 0 \tag{9}$$

or

$$\frac{\partial U^1}{\partial \xi^1} + \frac{\partial U^2}{\partial \xi^2} + \frac{\partial U^3}{\partial \xi^3} - U^1 \frac{\epsilon\lambda}{r_c} + U^2 \frac{1}{r_c} + O\left(\epsilon^2\right) = 0 \tag{10}$$

5.4 Kinematic boundary condition

The kinematic boundary condition can be derived as

$$\frac{\partial H}{\partial t} = U^3 - U^1 \frac{\partial H}{\partial \xi^1} - U^2 \frac{\partial H}{\partial \xi^2} \tag{11}$$

Next, let us consider the vertical integration of continuity

$$\int_b^H \left\{ \frac{\partial U^1}{\partial \xi^1} + \frac{\partial U^2}{\partial \xi^2} + \frac{\partial u_z}{\partial z} - U^1 \frac{\epsilon\lambda}{r_c} + U^2 \frac{1}{r_c}\right\} dz = 0 \tag{12}$$

or

$$\int_b^H \frac{\partial U^1}{\partial \xi^1} dz + \int_b^H \frac{\partial U^2}{\partial \xi^2} dz + \int_b^H \frac{\partial u_z}{\partial z} dz - \int_b^H U^1 \frac{\epsilon\lambda}{r_c} dz + \int_b^H U^2 \frac{1}{r_c} dz = 0 \tag{13}$$

Applying Leibnitz rule

$$\begin{aligned} &\frac{\partial}{\partial \xi^1}\int_b^H U^1 dz - \left(U^1 \frac{\partial H}{\partial \xi^1}\right)_{z=H} + \left(U^1 \frac{\partial b}{\partial \xi^1}\right)_{z=b} \\ &+ \frac{\partial}{\partial \xi^2}\int_b^H U^2 dz - \left(U^2 \frac{\partial H}{\partial \xi^2}\right)_{z=H} + \left(U^2 \frac{\partial b}{\partial \xi^2}\right)_{z=b} \\ &+ \frac{\partial}{\partial z}\int_b^H u_z dz - \left(u_z \frac{\partial H}{\partial z}\right)_{z=H} + \left(u_z \frac{\partial b}{\partial z}\right)_{z=b} \\ &- \frac{\epsilon\lambda}{r_c}\int_b^H U^1 dz + \frac{1}{r_c}\int_b^H U^2 dz = 0 \end{aligned} \tag{14}$$

or

$$\begin{aligned} &\frac{\partial}{\partial \xi^1}\int_b^H U^1 dz - \left(U^1 \frac{\partial H}{\partial \xi^1}\right)_{z=H} + \left(U^1 \frac{\partial b}{\partial \xi^1}\right)_{z=b} \\ &+ \frac{\partial}{\partial \xi^2}\int_b^H U^2 dz - \left(U^2 \frac{\partial H}{\partial \xi^2}\right)_{z=H} + \left(U^2 \frac{\partial b}{\partial \xi^2}\right)_{z=b} \\ &+ \left(u_z\right)_{z=H} - \left(u_z\right)_{z=b} - \frac{\epsilon\lambda}{r_c}\int_b^H U^1 dz + \frac{1}{r_c}\int_b^H U^2 dz = 0 \end{aligned} \tag{15}$$

or

$$\begin{aligned}(u_z)_{z=H} \quad - \quad & \left(U^1 \frac{\partial H}{\partial \xi^1}\right)_{z=H} \quad - \quad \left(U^2 \frac{\partial H}{\partial \xi^2}\right)_{z=H} \\ = \quad & - \frac{\partial}{\partial \xi^1} \int_b^H U^1 dz \; - \; \left(U^1 \frac{\partial b}{\partial \xi^1}\right)_{z=b} \; - \; \frac{\partial}{\partial \xi^2} \int_b^H U^2 dz \; - \; \left(U^2 \frac{\partial b}{\partial \xi^2}\right)_{z=b} \\ & + \; (u_z)_{z=b} \; + \; \frac{\epsilon\lambda}{r_c} \int_b^H U^1 dz \; - \; \frac{1}{r_c} \int_b^H U^2 dz \end{aligned} \tag{16}$$

so

$$\begin{aligned}\frac{\partial H}{\partial t} \quad = \quad & - \frac{\partial}{\partial \xi^1} \int_b^H U^1 dz \; - \; \left(U^1 \frac{\partial b}{\partial \xi^1}\right)_{z=b} \; - \; \frac{\partial}{\partial \xi^2} \int_b^H U^2 dz \; - \; \left(U^2 \frac{\partial b}{\partial \xi^2}\right)_{z=b} \\ & + \; (u_z)_{z=b} \; + \; \frac{\epsilon\lambda}{r_c} \int_b^H U^1 dz \; - \; \frac{1}{r_c} \int_b^H U^2 dz \end{aligned} \tag{17}$$

if all velocities are 0 at $z = b$ then

$$\frac{\partial H}{\partial t} \quad = \quad - \frac{\partial}{\partial \xi^1} \int_b^H U^1 dz \; - \; \frac{\partial}{\partial \xi^2} \int_b^H U^2 dz \; + \; \frac{\epsilon\lambda}{r_c} \int_b^H U^1 dz \; - \; \frac{1}{r_c} \int_b^H U^2 dz \tag{18}$$

A discrete form might be written as

$$\begin{aligned}\left(\frac{\partial H}{\partial t}\right)_{i,j} \quad = \quad & - \frac{1}{\Delta x} \left(\int_b^H U^1 dz\right)_{i+1/2,j} \; + \; \frac{1}{\Delta x} \left(\int_b^H U^1 dz\right)_{i-1/2,j} \\ & - \frac{1}{\Delta y} \left(\int_b^H U^2 dz\right)_{i,j+1/2} \; + \; \frac{1}{\Delta y} \left(\int_b^H U^2 dz\right)_{i,j-1/2} \\ & + \left(\frac{\epsilon\lambda}{2r_c} \int_b^H U^1 dz\right)_{i+1/2,j} \; + \; \left(\frac{\epsilon\lambda}{2r_c} \int_b^H U^1 dz\right)_{i-1/2,j} \\ & - \left(\frac{1}{2r_c} \int_b^H U^2 dz\right)_{i,j+1/2} \; - \; \left(\frac{1}{2r_c} \int_b^H U^2 dz\right)_{i,j-1/2} \end{aligned} \tag{19}$$

or

$$\begin{aligned}\left(\frac{\partial H}{\partial t}\right)_{i,j} \quad = \quad & - \frac{1}{\Delta x} \left\{\left(1 - \frac{\epsilon\lambda\Delta x}{2r_c}\right) \int_b^H U^1 dz\right\}_{i+1/2,j} \; + \; \frac{1}{\Delta x} \left\{\left(1 + \frac{\epsilon\lambda\Delta x}{2r_c}\right) \int_b^H U^1 dz\right\}_{i-1/2,j} \\ & - \frac{1}{\Delta y} \left\{\left(1 + \frac{\Delta y}{2r_c}\right) \int_b^H U^2 dz\right\}_{i,j+1/2} \; + \; \frac{1}{\Delta y} \left\{\left(1 - \frac{\Delta y}{2r_c}\right) \int_b^H U^2 dz\right\}_{i,j-1/2} \end{aligned} \tag{20}$$

Let

$$\alpha_{i,j} \quad = \quad \left(1 - \frac{\epsilon\lambda\Delta x}{2r_c}\right)_{i+\frac{1}{2},j} \tag{21}$$

$$\beta_{i,j} \quad = \quad \left(1 + \frac{\epsilon\lambda\Delta x}{2r_c}\right)_{i-\frac{1}{2},j} \tag{22}$$

$$\psi_{i,j} \quad = \quad \left(1 + \frac{\Delta y}{2r_c}\right)_{i,j+\frac{1}{2}} \tag{23}$$

$$\kappa_{i,j} \quad = \quad \left(1 - \frac{\Delta y}{2r_c}\right)_{i,j-\frac{1}{2}} \tag{24}$$

This provides

$$\left(\frac{\partial H}{\partial t}\right)_{i,j} = -\left(\frac{\alpha}{\Delta x}\int_b^H U^1 dz\right)_{i+1/2,j} + \left(\frac{\beta}{\Delta x}\int_b^H U^1 dz\right)_{i-1/2,j} - \left(\frac{\psi}{\Delta y}\int_b^H U^2 dz\right)_{i,j+1/2} + \left(\frac{\kappa}{\Delta y}\int_b^H U^2 dz\right)_{i,j-1/2} \tag{25}$$

Let the velocity be written in a matrix form (Casulli and Cheng, 1992) as

$$\mathbf{A}^n_{i+\frac{1}{2},j}\mathbf{U}^{n+1}_{i+\frac{1}{2},j} = \mathbf{G}^n_{i+\frac{1}{2},j} - g\frac{\Delta t}{\Delta x}\,\delta H^{n+1}\mathbf{\Delta Z}^n_{i+\frac{1}{2},j} \tag{26}$$

we then arrive at

$$\begin{aligned}\left(\frac{\partial H}{\partial t}\right)_{i,j} = & -\left[\frac{\alpha}{\Delta x}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i+1/2,j} + \left[\frac{\beta}{\Delta x}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i-1/2,j} \\ & + g\left[\alpha\frac{\Delta t}{\Delta x^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i+1/2,j}\delta H^{n+1}_{i+1/2,j} \\ & - g\left[\beta\frac{\Delta t}{\Delta x^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i-1/2,j}\delta H^{n+1}_{i-1/2,j} \\ & - \left[\frac{\psi}{\Delta y}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i,j+1/2} + \left[\frac{\kappa}{\Delta y}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i,j-1/2} \\ & + g\left[\psi\frac{\Delta t}{\Delta y^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i,j+1/2}\delta H^{n+1}_{i,j+1/2} \\ & - g\left[\kappa\frac{\Delta t}{\Delta y^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i,j-1/2}\delta H^{n+1}_{i,j-1/2}\end{aligned} \tag{27}$$

Expanding

$$\begin{aligned}\left(\frac{\partial H}{\partial t}\right)_{i,j} = & -\left[\frac{\alpha}{\Delta x}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i+1/2,j} + \left[\frac{\beta}{\Delta x}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i-1/2,j} \\ & + g\left[\alpha\frac{\Delta t}{\Delta x^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i+1/2,j}H^{n+1}_{i+1,j} \\ & - g\left[\alpha\frac{\Delta t}{\Delta x^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i+1/2,j}H^{n+1}_{i,j} \\ & - g\left[\beta\frac{\Delta t}{\Delta x^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i-1/2,j}H^{n+1}_{i,j} \\ & + g\left[\beta\frac{\Delta t}{\Delta x^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i-1/2,j}H^{n+1}_{i-1,j} \\ & - \left[\frac{\psi}{\Delta y}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i,j+1/2} + \left[\frac{\kappa}{\Delta y}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i,j-1/2} \\ & + g\left[\psi\frac{\Delta t}{\Delta y^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i,j+1/2}H^{n+1}_{i,j+1} \\ & - g\left[\psi\frac{\Delta t}{\Delta y^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i,j+1/2}H^{n+1}_{i,j} \\ & - g\left[\kappa\frac{\Delta t}{\Delta y^2}\left(1-\beta_{i,j}\right)\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i,j-1/2}H^{n+1}_{i,j}\end{aligned}$$

$$+ \, g\left[\kappa\frac{\Delta t}{\Delta y^2}\ \left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i,j-1/2} H^{n+1}_{i,j-1} \tag{28}$$

Grouping

$$\begin{aligned}
H^{n+1}_{i,j} \ - \ H^n_{i,j} \ = \ H^{n+1}_{i,j}\Bigg\{ & - g\left[\alpha\frac{\Delta t^2}{\Delta x^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i+1/2,j} \\
& - g\left[\beta\frac{\Delta t^2}{\Delta x^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i-1/2,j} \\
& - g\left[\psi\frac{\Delta t^2}{\Delta y^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i,j+1/2} \\
& - g\left[\kappa\frac{\Delta t^2}{\Delta y^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i,j-1/2}\Bigg\} \\
& - \left[\alpha\frac{\Delta t}{\Delta x}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i+1/2,j} + \left[\beta\frac{\Delta t}{\Delta x}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i-1/2,j} \\
& + g\left[\alpha\frac{\Delta t^2}{\Delta x^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i+1/2,j} H^{n+1}_{i+1,j} \\
& + g\left[\beta\frac{\Delta t^2}{\Delta x^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i-1/2,j} H^{n+1}_{i-1,j} \\
& - \left[\psi\frac{\Delta t}{\Delta y}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i,j+1/2} + \left[\kappa\frac{\Delta t}{\Delta y}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i,j-1/2} \\
& + g\left[\psi\frac{\Delta t^2}{\Delta y^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i,j+1/2} H^{n+1}_{i,j+1} \\
& + g\left[\kappa\frac{\Delta t^2}{\Delta y^2}\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i,j-1/2} H^{n+1}_{i,j-1}
\end{aligned} \tag{29}$$

Let

$$s^n_{i\pm\frac{1}{2},j} \ = \ g\frac{\Delta t^2}{\Delta x^2}\left[\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i\pm\frac{1}{2},j} \tag{30}$$

$$s^n_{i,j\pm\frac{1}{2}} \ = \ g\frac{\Delta t^2}{\Delta y^2}\left[\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{\Delta Z}\right]^n_{i,j\pm\frac{1}{2}} \tag{31}$$

$$\begin{aligned}
d^n_{i,j} \ &= \ 1 + (\alpha\, s^n)_{i+\frac{1}{2},j} + (\beta\, s^n)_{i-\frac{1}{2},j} + (\psi\, s^n)_{i,j+\frac{1}{2}} + (\kappa\, s^n)_{i,j-\frac{1}{2}} \\
q^n_{i,j} \ &= \ H^n_{i,j} - \frac{\Delta t}{\Delta x}\left[\alpha\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i+1/2,j} + \frac{\Delta t}{\Delta x}\left[\beta\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i-1/2,j} \\
& \quad - \frac{\Delta t}{\Delta y}\left[\psi\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i,j+1/2} + \frac{\Delta t}{\Delta y}\left[\kappa\left(\mathbf{\Delta Z}\right)^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{G}\right]^n_{i,j-1/2}
\end{aligned} \tag{32}$$

So we have

$$\begin{aligned}
d^n_{i,j}H^{n+1}_{i,j} \ & - \ (\alpha\, s^n)_{i+\frac{1}{2},j}\, H^{n+1}_{i+1,j} - (\beta\, s^n)_{i-\frac{1}{2},j}\, H^{n+1}_{i-1,j} \\
& - \ (\psi\, s^n)_{i,j+\frac{1}{2}}\, H^{n+1}_{i,j+1} - (\kappa\, s^n)_{i,j-\frac{1}{2}}\, H^{n+1}_{i,j-1} \ = \ q^n_{i,j}
\end{aligned} \tag{33}$$

The normalized form

$$\sqrt{d^n_{i,j}}H^{n+1}_{i,j} \ - \ \frac{(\alpha\, s^n)_{i+\frac{1}{2},j}}{\sqrt{d^n_{i,j}d^n_{i+1,j}}}\sqrt{d^n_{i+1,j}}H^{n+1}_{i+1,j}$$

$$
\begin{aligned}
&- \frac{(\beta\, s^n)_{i-\frac{1}{2},j}}{\sqrt{d^n_{i,j} d^n_{i-1,j}}} \sqrt{d^n_{i-1,j}} H^{n+1}_{i-1,j} \\
&- \frac{(\psi\, s^n)_{i,j+\frac{1}{2}}}{\sqrt{d^n_{i,j} d^n_{i,j+1}}} \sqrt{d^n_{i,j+1}} H^{n+1}_{i,j+1} \\
&- \frac{(\kappa\, s^n)_{i,j-\frac{1}{2}}}{\sqrt{d^n_{i,j} d^n_{i,j-1}}} \sqrt{d^n_{i,j-1}} H^{n+1}_{i,j-1} \;=\; q^n_{i,j} \qquad (34)
\end{aligned}
$$

define

$$
\begin{aligned}
e_{i,j} &= \sqrt{d^n_{i,j}}\, H^{n+1}_{i,j} && (35)\\
a_{i\pm\frac{1}{2},j} &= \frac{s^n_{i\pm\frac{1}{2},j}}{\sqrt{d^n_{i,j} d^n_{i\pm1,j}}} && (36)\\
a_{i,j\pm\frac{1}{2}} &= \frac{s^n_{i,j\pm\frac{1}{2}}}{\sqrt{d^n_{i,j} d^n_{i,j\pm1}}} && (37)\\
b_{i,j} &= \frac{q^n_{i,j}}{\sqrt{d^n_{i,j}}} && (38)
\end{aligned}
$$

We have

$$
\begin{aligned}
e_{i,j} \;&-\; (\alpha\, a)_{i+\frac{1}{2},j}\, e_{i+1,j} \\
&-\; (\beta\, a)_{i-\frac{1}{2},j}\, e_{i-1,j} \\
&-\; (\psi\, a)_{i,j+\frac{1}{2}}\, e_{i,j+1} \\
&-\; (\kappa\, a)_{i,j-\frac{1}{2}}\, e_{i,j-1} \;=\; b_{i,j} \qquad (39)
\end{aligned}
$$

Chapter 6

Straightening the bathymetry with cubic splines

We generally have available a bathymetry data set that provides the depth (d) as $d = f(x_k, y_k)$ over some set of k points[1]. We'd like to compute a corresponding curvilinear system of $d\left(\xi^1, \xi^2\right)$ that exists on some regular rectangular array of $m \times n$ grid cells. If we select a discrete set of points $(x_i,\, y_i)$ for $i = 1, n$ in physical space that represent the approximate center (or "thalweg") of the river or estuary[2], we can compute a continuous representation of the centerline in terms of cubic splines of the form

$$y \;=\; a\,(x - x_{lo})^3 \;+\; b\,(x - x_{lo})^2 \;+\; c\,(x - x_{lo}) \;+\; d \tag{1}$$

where $x_{lo} \le x \le x_{hi}$ and $x_{lo},\, x_{hi}$ are a pair of $x_i,\, x_{i+1}$ points in the discrete set. As the thalweg of a river or estuary is may *not* be single-valued in either the x or y coordinate system, it is necessary to divide the thalweg up into discrete sections that can be represented by separate splines in with the x or y axis as the independent axis as necessary. The cubic spline coefficients are readily computed using methods in any number of textbooks (e.g. Al-Khafaji and Tooley, 1986). The relationship between the x, y and the arclength s is given by

$$ds^2 \;=\; dx^2 \;+\; dy^2 \tag{2}$$

We can write

$$\left(\frac{ds}{dx}\right)^2 \;=\; \left(\frac{dx}{dx}\right)^2 \;+\; \left(\frac{dy}{dx}\right)^2 \tag{3}$$

For a general cubic equation of the form of equation (1) we have

$$\frac{dy}{dx} \;=\; 3a\,(x - x_{lo})^2 \;+\; 2b\,(x - x_{lo}) \;+\; c \tag{4}$$

from which it follows that

$$\left(\frac{ds}{dx}\right)^2 \;=\; 1 \;+\; \left[3a\,(x - x_{lo})^2 \;+\; 2b\,(x - x_{lo}) \;+\; c\right]^2 \tag{5}$$

[1] The bathymetry data set is often a rectangular array of regularly-spaced data, but need not be limited to such a set in the following work.

[2] The thalweg points are *not* required to be a subset of $(x_k,\, y_k)$ points used in the bathymetry, although it is convenient to specify them as such.

The arc length from $x = x_{lo}$ to any point $x_{lo} \leq x \leq x_{hi}$ along the cubic spline can be found from numerically solving the differential equation:

$$\frac{ds}{dx} = \left\{ 1 + \left[3a\ (x - x_{lo})^2 + 2b\ (x - x_{lo}) + c \right]^2 \right\}^{1/2} \quad : \quad x_{lo} \leq x \leq x_{hi} \tag{6}$$

So the arc length at any point x, y on the thalweg can be found by successively solving the above differential equation over each cubic spline section. This allows us to develop an algorithm for computing a set of discrete (x, y) locations along the thalweg that are evenly spaced along the arc length. The computation of the slope of the tangent and normal lines at any point along a cubic spline curve is trivial as the tangent slope is simply equation (4) and the normal slope is the negative inverse of the tangent slope. One can readily compute a discrete set of points that are uniformly spaced along the normal line. As long as $r/r_c < 1$ the normal lines will not cross and every discrete point in the domain will be simply-connected.

The radius of curvature (r_c) at any point along a 2D curve is readily found as (Borisenko and Tarapov, 1968)

$$r_c = \left\{ \left(\frac{d^2x}{ds^2} \right)^2 + \left(\frac{d^2y}{ds^2} \right)^2 \right\}^{-1/2} \tag{7}$$

This gives us the tools to compute the values of ϵ, λ and r_c^{-1} throughout the domain, providing the necessary coefficients for computing either (1) the neglected terms in a "straightened" solution or (2) adding the order ϵ or higher terms to the model equations.

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