Rivers and Electric Networks: Crossing Disciplines in Modeling and Simulation

Ben R. Hodges
University of Texas at Austin
Center for Research in Water Resources,
The University of Texas at Austin,
10100 Burnet Road, Bldg. 119, Stop R8000
Austin, TX 78758-4445, USA
hodges@utexas.edu

Frank Liu
IBM Research Austin
11501 Burnet Road,
Austin, TX 78758, USA
frankliu@us.ibm.com
## Contents

1 Introduction 2

1.1 Crossing disciplines is not always interdisciplinary . . . . . 2
1.2 Networks . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
1.3 Notation . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
1.4 Rivers and electric circuits . . . . . . . . . . . . . . . . . 10
1.5 The development of river hydraulics . . . . . . . . . . . . 14
1.6 Summary . . . . . . . . . . . . . . . . . . . . . . . . . . . 18

2 Electric Circuit Networks 19

2.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . 19
2.2 Maxwell’s equations . . . . . . . . . . . . . . . . . . . . . 20
2.3 Quasi-static approximation for circuit analysis . . . . . . . 22
2.4 Models of different lumped devices . . . . . . . . . . . . . 27
2.5 Circuit network analysis based on Kirchhoff’s laws . . . . . 33
2.6 The role of data in electric circuits . . . . . . . . . . . . . 37
2.7 Summary . . . . . . . . . . . . . . . . . . . . . . . . . . . 40

3 Governing Equations for River Flow 41

3.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . 41
3.2 Three-dimensional equations . . . . . . . . . . . . . . . . . 42
3.3 Reynolds-averaged Navier-Stokes equations . . . . . . . . 44
<table>
<thead>
<tr>
<th>3.4 Channel curvature</th>
<th>46</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5 Cross-section integration</td>
<td>47</td>
</tr>
<tr>
<td>3.6 Hydrostatic approximation</td>
<td>50</td>
</tr>
<tr>
<td>3.7 Traditional 1D Saint-Venant equations</td>
<td>51</td>
</tr>
<tr>
<td>3.8 Friction slope</td>
<td>54</td>
</tr>
<tr>
<td>3.9 Summary</td>
<td>55</td>
</tr>
<tr>
<td>4 A River Network Primer</td>
<td>57</td>
</tr>
<tr>
<td>4.1 Introduction</td>
<td>57</td>
</tr>
<tr>
<td>4.2 Boundary conditions</td>
<td>58</td>
</tr>
<tr>
<td>4.3 Normal depth and initial conditions</td>
<td>59</td>
</tr>
<tr>
<td>4.4 Observed water elevation and flow relationships</td>
<td>60</td>
</tr>
<tr>
<td>4.5 Energy description of flow</td>
<td>61</td>
</tr>
<tr>
<td>4.6 Flood waves and backwater waves</td>
<td>65</td>
</tr>
<tr>
<td>4.7 Subcritical and supercritical flow</td>
<td>67</td>
</tr>
<tr>
<td>4.8 Cross-section Surveys</td>
<td>68</td>
</tr>
<tr>
<td>4.9 Unsurveyed rivers</td>
<td>70</td>
</tr>
<tr>
<td>4.10 Tributary junctions</td>
<td>71</td>
</tr>
<tr>
<td>4.11 Dams, lakes, and hydraulic structures</td>
<td>73</td>
</tr>
<tr>
<td>4.12 Floodplains</td>
<td>79</td>
</tr>
<tr>
<td>4.13 Connecting with estuaries</td>
<td>80</td>
</tr>
<tr>
<td>4.14 Summary</td>
<td>80</td>
</tr>
<tr>
<td>5 Numerical Connections between Electric and River Networks</td>
<td>82</td>
</tr>
<tr>
<td>5.1 Introduction</td>
<td>82</td>
</tr>
<tr>
<td>5.2 River channel PDE's</td>
<td>83</td>
</tr>
<tr>
<td>5.3 Discretization scheme</td>
<td>84</td>
</tr>
<tr>
<td>5.4 Boundary conditions</td>
<td>86</td>
</tr>
<tr>
<td>5.5 Numerical techniques</td>
<td>88</td>
</tr>
<tr>
<td>5.6 Smoothness of equivalent friction geometry</td>
<td>95</td>
</tr>
<tr>
<td>5.7 Numerical examples</td>
<td>98</td>
</tr>
<tr>
<td>5.8 Summary</td>
<td>100</td>
</tr>
<tr>
<td>6 Final Remarks</td>
<td>102</td>
</tr>
<tr>
<td>6.1 A reconnection of disciplines</td>
<td>102</td>
</tr>
<tr>
<td>6.2 Similarities and contrasts</td>
<td>104</td>
</tr>
</tbody>
</table>
iv

6.3 Future paths ................................. 106

Acknowledgements .............................. 109
References ...................................... 110
Abstract

Electric circuits and river networks share similarities in both their network structure and derivation from conservation principals. However, the disciplines have evolved separately and developed methods and models. This paper presents the foundations for network analysis for both disciplines and shows how numerical methods developed for circuit simulations can significantly improve river network models. The equations, models, and jargon are described, providing a reference for future studies to transfer knowledge across disciplinary boundaries.


DOI: 10.1561/1000000033.
1

Introduction

1.1 Crossing disciplines is not always interdisciplinary

Interdisciplinary studies typically occur at the boundary between closely related areas, for example VLSI (Very Large Scale Integration) design meets at the boundaries of material science, electrical engineering, and computer science; similarly, water quality modeling depends on chemistry, biology, and physical transport processes. The importance of breaking through disciplinary isolation has been well-recognized for the purpose of addressing such problems. Less recognized are the opportunities to apply established ideas from one discipline in another, seemingly unrelated, discipline. Indeed, there is a significant challenge here: how do you move ideas across a boundary when there is no problem at the boundary and no link between the disciplines? We might call this the “cross-disciplinary” problem. In the present work we are interested in river networks – at the largest scale covering continents – that do not share any interdisciplinary boundaries with microscopic electronic circuits in semiconductors. The only connection is that both are network problems.

As opposed to interdisciplinary knowledge transfer, cross-disciplinary transfer tends to be serendipitous rather than organized,
1.1. Crossing disciplines is not always interdisciplinary

typically an unexpected result of curiosity. Indeed, unlike interdisciplinary studies, there do not appear to be any formal programs for fostering cross-disciplinary knowledge transfer. It seems that “thinking outside the box” mostly means working at the boundaries of your box, rather than stepping into another completely different box. Cross-disciplinary transfer can also be a one-way affair; i.e. the methods/ideas from one discipline may provide immediate improvement or advancement in another, but reverse transfers might be obscure or entirely non-existent. From the authors’ experience there are three key challenges to crossing disciplines: (1) identifying the opportunities where one discipline has advances that might be useful to another, (2) communicating across the jargon-laced literature of different disciplines, and (3) clearly articulating and demonstrating the benefits and value to broader research communities.

To date, our efforts in crossing disciplines have been one-way: using insights from electric circuits to improve river models. Because of non-linear complexities in governing equations, the development path for river modeling diverged from that of electric circuit modeling. In river modeling, approximations were made to fit computer and numerical capabilities in the 1970s and 1980s and were never significantly revisited – despite increasing computer power and improved numerical methods. Numerical methods developed in the 1990s and 2000s to solve microchip circuit simulation problems in VLSI simply never reached river network modelers. The underlying cause is likely the disparity in commercial markets, which are limited for river models and give little encouragement to investing in model updates; indeed, some of the most commonly used models are built on numerical frameworks developed in government-funded projects in the 1970s. Remedying these problems was the focus of [52], which provided a cross-disciplinary experience that inspired the present paper. Although we are writing from the foundations of our electric-to-river experience, it is hoped we might inspire future synergies that cross boundaries back in the other direction.

This paper provides the foundations for communication between electrical and hydraulic engineers/scientists who work on network prob-
lems. The audience is intended to be from both disciplines, so we endeavor to explain the physics of both systems and draw out their similarities and differences. In the process, we present an example of how numerical methods from microchip circuit analysis have been used in modeling a river network to a level of detail that previously was not possible. We are not attempting to provide a rigorous survey of all the work in these two areas as the background literature is simply too vast; the citations herein should be considered examples rather than exhaustive. Indeed, our focus is on putting forward what are mostly textbook ideas in a format that is accessible across boundaries.

Our introduction begins with an overview of the key similarities and differences between rivers and electric circuits (§1.2–1.4), which introduces the reader to the jargon and relationships between disciplines. Interestingly, because today’s general physics curriculum typically introduces electric circuits but not open-channel flow, hydraulic engineers might have a better foundation in electrical jargon than vice-versa. Unfortunately, a major impediment to quickly understanding hydraulics is that the jargon is rooted in history. Because civil engineering projects are long-lived and costly the discipline is slow to change; there are curiosities in the jargon and equations that must be accepted simply as echoes of history. With this problem in mind, we provide a short introduction on how river hydraulics developed (§1.5), which will help the electrical engineering reader understand not just what the hydraulic equations are, but why they take their peculiar forms.

In keeping with our goal of introducing the disciplines across boundaries, beyond this introduction the reader will find a brief review of electric theory (Chapter 2), which is intended as both a primer for hydraulic engineers and to provide analogies between circuits and rivers to guide electrical engineers through the in-depth discussion of the governing equations for river flow in a single channel (Chapter 3) and the peculiarities of river networks (Chapter 4). These sections on theory are followed by a discussion of numerical approaches that have been applied crossing disciplines (Chapter 5). We close with some final thoughts on the similarities and contrasts between these disciplines, and possible areas for future work (Chapter 6).
1.2 Networks

The common feature of large-scale river networks, e.g. Fig. 1.1, and VLSI networks, e.g. Fig. 1.2, is the complexity of their interconnections. At the most basic level, river networks can be treated as directed acyclic graphs (DAG) connecting tributary junctions with differential equations representing the evolution of water surface elevation and flow rate. River systems are inherently acyclic because water must flow downhill; i.e. without a pump it is impossible for water to return to an uphill point to form a cyclic graph. In contrast, the network structure for a VLSI can include cyclic connections. Furthermore, rivers are mostly simple tree structures as upstream tributary branches join to downstream main-stem channels in the classic root-branch pattern (Fig. 1.1). However, important exceptions occur – particularly where water flows around a mid-channel island (Figs. 1.3, 1.4), where man-made canals create acyclic paths (Fig. 1.5), or the distributaries where a river debouches into the ocean in a network of channels that branch and join through a complex maze of marshes (Fig. 1.6).

Figure 1.1: Amazon River basin. URL: http://daac.ornl.gov/LBA/guides/CD06_CAMREX.html
Introduction

Figure 1.2: Connection network of a VLSI chip. URL: http://vlsicad.eecs.umich.edu/BK/FGR/. Reproduced with permission

Figure 1.3: Charley River at Yukon showing mid-channel islands (Photo by USGS) URL: http://ak.water.usgs.gov/yukon/index.php
Figure 1.4: Niagara River and Grand Island from Lake Erie (south) to Niagara Falls (USA/Canada) illustrating river split and rejoining. GoogleEarth, image ©USGS and DigitalGlobe
Figure 1.5: Atchafalaya River and canals near Morgan City (Louisiana, USA). GoogleEarth, image ©TerraMetrics

Figure 1.6: Wax Lake Delta (Louisiana, USA). GoogleEarth, image ©TerraMetrics
1.3 Notation

Cross-disciplinary research in any area is particularly challenging due to the difference in jargon and notation. To facilitate the readers from diverse backgrounds, Table 1.1 provides some of the common variables in river hydraulics, and Table 1.2 provides the same for electric circuits.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>velocity non-uniformity coefficient</td>
<td>–</td>
</tr>
<tr>
<td>$\eta$</td>
<td>free surface elevation</td>
<td>m</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>kinematic viscosity</td>
<td>m$^2$/s</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>cross-section breadth</td>
<td>m$^2$/s</td>
</tr>
<tr>
<td>$a$</td>
<td>wetted cross-sectional area</td>
<td>m$^2$</td>
</tr>
<tr>
<td>$g$</td>
<td>gravity</td>
<td>m/s$^2$</td>
</tr>
<tr>
<td>$h$</td>
<td>depth</td>
<td>m</td>
</tr>
<tr>
<td>$L$</td>
<td>length</td>
<td>m</td>
</tr>
<tr>
<td>$n$</td>
<td>Manning’s n</td>
<td>see §1.5.2</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure</td>
<td>Pa</td>
</tr>
<tr>
<td>$P$</td>
<td>time-averaged pressure</td>
<td>Pa</td>
</tr>
<tr>
<td>$q$</td>
<td>inflow rate per unit length</td>
<td>m$^2$/s</td>
</tr>
<tr>
<td>$Q$</td>
<td>volume flow rate</td>
<td>m$^3$/s</td>
</tr>
<tr>
<td>$R_h$</td>
<td>hydraulic radius</td>
<td>m</td>
</tr>
<tr>
<td>$S$</td>
<td>water surface slope</td>
<td>–</td>
</tr>
<tr>
<td>$S_f$</td>
<td>friction slope</td>
<td>–</td>
</tr>
<tr>
<td>$S_0$</td>
<td>slope of channel bottom</td>
<td>–</td>
</tr>
<tr>
<td>$u$</td>
<td>flow velocity</td>
<td>m/s</td>
</tr>
<tr>
<td>$U$</td>
<td>time-averaged flow velocity</td>
<td>m/s</td>
</tr>
<tr>
<td>$z$</td>
<td>vertical elevation</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 1.1: Common variables in river hydraulics
Introduction

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>conductance</td>
<td>siemens (S)</td>
</tr>
<tr>
<td>$I$</td>
<td>electric current</td>
<td>ampere (A)</td>
</tr>
<tr>
<td>$L$</td>
<td>inductance</td>
<td>henry (H)</td>
</tr>
<tr>
<td>$R$</td>
<td>resistance</td>
<td>ohm (Ω)</td>
</tr>
<tr>
<td>$Q$</td>
<td>electric charge</td>
<td>coulomb (C)</td>
</tr>
<tr>
<td>$V$</td>
<td>voltage</td>
<td>volt (V)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>conductivity</td>
<td>$S \cdot m^{-1}$</td>
</tr>
</tbody>
</table>

Table 1.2: Common variables in electric circuits

1.4 Rivers and electric circuits

River networks develop because water flows downhill; more precisely, the potential energy difference between higher and lower elevations causes water to seek a lower energy state. The elevation gradients of a river bed give rise to hydrostatic pressure differences that are effectively the voltages driving the river current. However as the landscape elevations cannot be reversed in the same way as an electric potential, a river flow is (for the most part) a unidirectional process, which makes river networks topologically simpler than electric circuits. With a clear path in an open channel, water seeks the lower potential energy state at a flow rate balancing the driving pressure gradient against frictional resistance. Dynamic river network models represent the flow rate and water surface elevation as functions of space and time.

The fundamental analogy for our work is that the water flow rate $Q$, in $m^2 s^{-1}$, is similar to electric current ($I$); the local water surface gradient $\partial \eta / \partial x$, provides a driving pressure gradient similar to a voltage between two terminals $V$; and water flow, like charge flow, is moderated by resistance. To illustrate this point, consider a simple resistive conductor versus a simple river channel, as shown in Fig. 1.7. For this conceptual model, the current ($I$) is caused by the voltage (or potential difference in the electric field) between two terminals of the conductor and is moderated by the resistance ($R$) in Ohm’s law as
1.4. Rivers and electric circuits

$I = V/R$. Similarly, for the river channel the water flow is caused by the potential difference in earth’s gravitational field associated with the difference water elevations ($\Delta \eta$), and moderated by frictional resistance $R$. This idea can be written as an idealized energy law in the form $Q^2 = g \Delta \eta / R$, where $g$ is gravitational acceleration.

Except in the superconductivity state, all conductors have resistivity. The resistance to charge flow (or its inverse, conductance) is a function of the cross-section area, material, and the element length. Similarly, the water flow resistance of some section of river channel depends on its physical characteristics: the cross-section area, the length of “wetted” perimeter where water contacts the river bed, the material of the river bed (e.g. sand, gravel, or boulders), the turbulence in the flow, and how all these physical characteristics change over the length.

![Figure 1.7: Current flow in an ideal resistor with rectangular cross section versus water flow in a rectangular channel](image)

This simplistic river-circuit analogy is useful for an intuitive understanding of relationships between the disciplines. However, for quantitative analysis of more realistic problems, we must face several confounding factors. Firstly, unlike electrons, water has significant mass and therefore inertia. Inertia provides flow “memory,” such that the recent past affects the near future flow evolution. As a result, simple models using proportional and instantaneous response to changing conditions (which are common in electric circuits) are poor representations of flow physics. Inertia is governed by Newton’s second law, $\sum F = ma$, so sudden introduction of an adverse pressure gradient (the equivalent of reversing $V$) requires time and distance to slow, stop and reverse the flow. The equivalent behavior in an electric circuit requires a component in which $\partial I / \partial t = f(I, V)$, where $f$ may be a nonlinear in the independent variables. To the first order, this behavior is similar
to an inductive element in a circuit, albeit more complicated since the underlying function can be nonlinear. This topic is discussed more fully in §2.4.

A second confounding factor is the complicated relationship between the water surface elevation and the driving pressure gradient in a river. To the first order, the pressure gradient is proportional to the gradient of the water surface elevation, i.e. $\partial \eta / \partial x$, where $\eta$ is the vertical distance of the water surface from a $z = 0$ baseline. To make another river-circuit analogy, we can imagine a circuit where a local voltage $V$ is generated as a gradient of some auxiliary variable ($\eta$), which is itself a function of current $I$ and another auxiliary variable representing effects of the conductor’s geometry ($\Gamma$). As another layer of complication, a change in water elevation will change a river’s cross-sectional area, so to continue the analogy we imagine an electric conductor that also changes geometry $\Gamma$ as a function of auxiliary variable $\eta$.

A third confounding factor is the relationship between resistance and flow in a river. The frictional resistance of a river is a strong function of both flow rate and geometry, whereas resistance in an electric conductor is strongly affected by geometry but only weakly affected by current – the latter typically through changes in material temperature when carrying large currents. Therefore, our river-circuit analogy requires a resistance $R$ that is a function of both $I$ and $\Gamma$.

To summarize, an electric circuit that has functional dependencies analogous to a river requires:

\[
\begin{align*}
\Gamma &= f_1 (\eta) \\
V &= f_2 (\eta, I, \Gamma) \\
R &= f_3 (I, \Gamma) \\
I &= f_4 (V, R, \Gamma)
\end{align*}
\]

where $V$, $R$, and $I$ are the traditional electric circuit variables, $\Gamma$ is geometry, and $\eta$ is an auxiliary function that couples geometry and potential. For rivers, the equivalent functions are typically nonlinear and require empirical coefficients that vary in different rivers – and often vary in different reaches of a single river. In contrast, simple
1.4. Rivers and electric circuits

electric circuit relationships typically follow:

\[ I = \frac{V}{R} \]  
\[ R = f(\Gamma) \]  
\[ \Gamma = \text{constant.} \]

where empirical coefficients are generally associated with material properties. Thus, key differences between electric circuits and river networks are in the complicated nonlinearities and dependencies between flow, the driving potential, and geometry associated with rivers.

The above analogy glosses over the central problem of fluid mechanics: uncertainties associated with turbulence. The functional dependence of turbulence on both flow rate and geometry adds empirical complexity to quantifying frictional losses (i.e. the \( R \) in an analogous electric circuit). In a river, the resistance depends on both the shape of the channel and the bed material in contact with the flow. Typically, we have incomplete data on both river geometry and bed material, which makes it difficult to parameterize flow resistance. Detailed bathymetric surveys are expensive and, to date, the only remote sensing technique that can directly measure river geometry is blue-green lidar, which requires clear water for effectiveness [40]. Furthermore, river channels are continually undergoing both erosion and aggradation, so even a perfect survey has only ephemeral validity. From satellite photogrammetry it is possible to define river widths and monitor water level changes with time [2, 31], but routinely quantifying the shape and composition of the channel bottom for a continental-scale network of more than \( 10^6 \) km of river requires surveying resources that are simply beyond practical capabilities. Surprisingly, lack of adequate data is also a problem in semiconductor circuits, where the average width of the conducting wires is less one micrometer (\( 10^{-6} \) m). However, this particular issue is partially addressed by macromodeling in electric circuit analysis, which we revisit in §2.6.

In summary, river networks are topologically simpler than electric circuit networks, but are governed by coupled nonlinear equations that are more complicated than the simplest electrical relationships. However, the more sophisticated devices that are common in semiconductor
circuitry allow a more complete, albeit more complicated, analogy between systems (see Chapter 2).

1.5 The development of river hydraulics

1.5.1 A brief history

Organized hydraulics can be traced back to the digging of irrigation channels in pre-historic Mesopotamia. As cities grew throughout the ancient world, the need for effective water supply, sanitation, and transportation resulted in dominant cities appearing on major rivers. For millennia, hydraulics principally advanced through slow accumulation of empirical knowledge, whose apex is arguably seen in the viaducts of the Roman Empire. An engineer from that era would have been unsurprised by the hydraulics of medieval Europe. This slow advance changed in the 18th century with the widespread dissemination of the calculus along with theoretical and experimental advances in fluid mechanics. A short history of the foundations of modern hydraulics from this era can be found in [24], from which we will extract some high points to provide context for some of the curious equation forms that are still used in river engineering.

Arguably, modern hydraulics begins with Antoine Chézy (1718-1798) and Pierre-Georges-Louis du Buat, who observed a relationship between flow velocity ($u$) and the channel slope ($S$) now known as the Chézy equation: $u = C\sqrt{Sa/P_w}$, where $a$ is the cross-sectional area, $P_w$ is the wetted perimeter of the channel (i.e. where water contacts the bed, see Fig. 1.8), and $C$ is an empirical coefficient that varies in different rivers, now known as the Chézy coefficient. Note that the $C$ is a “conveyance” coefficient in that increasing values result in increasing velocity (and hence flow rate for a given $a$). Most modern hydraulic equations use a “roughness” coefficient – essentially an inversion of $C$ – such that increasing values cause decreasing flow rates. Roughness and conveyance are similar to inverse concepts of resistivity and conductivity in electric circuits. In later notation, the “hydraulic radius”
(\(R_h\)) was introduced into the Chézy equation as

\[
R_h \equiv \frac{a}{P_w}
\]  

so \(u = C\sqrt{R_h S}\). Although the 18\(^{th}\) century Chézy equation is rarely used today, the hydraulic radius is still encountered as a key geometry definition in many 20\(^{th}\) and 21\(^{st}\) century hydraulic models.

![Illustrative example of a trapezoidal cross section](image)

**Figure 1.8:** Illustrative example of a trapezoidal cross section, as often used by hydraulic engineers in viaducts for water transport. A natural river cross section is much more complicated.

In the 19\(^{th}\) and early 20\(^{th}\) centuries, the Chézy approach was modified and eventually supplanted by Manning’s equation (§1.5.2) for steady flow, while on a separate path Newton’s equations of motion were developed into the three-dimensional (3D) Navier-Stokes equations (§3.2) and the 1D Saint-Venant equations (§1.5.3).

### 1.5.2 Chezy-Manning equation

In the mid 1800s, experimentalists Darcy and Bazin provided data from which Phillippe Gauckler (1826-1905) observed that Chézy’s \(C\) could be further refined by using \(u = \alpha R_h^{2/3} S^{1/2}\) for \(S > 7 \times 10^{-4}\) or \(u = \beta R_h^{3/4} S\) for \(S < 7 \times 10^{-4}\); where \(\alpha, \beta\) are empirical coefficients. Gauckler put forward his formula in 1868 [33], but his place in history
was supplanted by Robert Manning, who published a more widely-read version of a similar equation in 1889 that usually bears his name. This formula is now generally written as

\[ u = \frac{1}{n} R_h^{2/3} S^{1/2} \]  

(1.9)

where \( n \) has a reciprocal relationship with Chézy’s \( C \) and Gauckler’s \([\alpha, \beta]\). Commonly known as “Manning’s \( n \)” in the English-speaking world, this coefficient is also misnamed: \( n \) was not in the original formula of Manning, but became a common adaptation to match the roughness \( n \) proposed by Wilhelm R. Kutter [24]. In the U.S. eq. (1.9) is often called the Chézy-Manning equation to acknowledge the contribution of Chézy in the \( 1/2 \) power relationship with the channel slope. In Europe the equation is known as Manning-Strickler or Gauckler-Manning-Strickler to acknowledge Albert Strickler, whose early 20th century work provided a rigorous approach to quantifying roughness [70].

Note that eq. (1.9) implies that \( n \) has units of \( TL^{-1/3} \), so that a common non-dimensional reformulation is to replace \( n^{-1} \) with \( Kn^{-1} \), so that \( n \) can be considered non-dimensional. The coefficient \( K \) takes on a value of unity for \([m, s]\) and \( K = 1.486 \) for units of \([ft, s]\). The \( K \) form is not consistently used in the literature, but implicitly hydraulic engineers treat \( n \) as non-dimensional and use either \( 1.486/n \) or \( 1/n \) along with consistent units for \( u \) and \( R_h \). Because \( n \) is generally not known with great precision, \( K = 1.49 \) or \( K = 1.5 \) are often seen for \([ft, s]\) units.

In river modeling, Manning’s equation is typically written in terms of the cross-section integrated volumetric flow rate, \( Q \) with units of \( m^3s^{-1} \) or \( ft^3s^{-1} \). Treating the \( u \) in eq. (1.9) as the average velocity, the flow rate for cross-sectional area of \( a \) is given by \( Q = ua \) so that

\[ Q = \frac{1}{n} a R_h^{2/3} S^{1/2} \]  

(1.10)

The above provides a simple approach for quantifying steady flow in rivers that are either “uniform” or “gradually varying” in space. In hydraulics jargon, a uniform flow does not have spatial gradients in geometry, \( S \), or \( n \), whereas a gradually-varying flow allows weak gradients of independent variables. Where sharp spatial changes in any
variable occur, Manning’s equation is invalid and the flow is considered “rapidly varying.” The use of “gradually” and “rapidly” to describe spatial gradients rather than temporal gradients is an unfortunate legacy in hydraulic engineering jargon that can only be defended by resorting to a Lagrangian viewpoint: when following a particle in a gradually- or rapidly-varying flow the observed temporal changes will indeed be either gradual or rapid, respectively.

Strictly speaking, the slope $S$ in the Chézy and Manning equations is the channel bottom slope, which is typically designated as $S_0$. If a long section of river is spatially uniform in both $S_0$ and channel geometry, then for steady flow (constant in time) the water surface slope will be exactly parallel to the channel bottom implying an identical slope. Such flows are said to be uniform and the “normal” flow (§4.3) will approximately satisfy eq. (1.10) using $S_0$. A more general form of Manning’s equation uses the “friction slope,” $S_f$ for $S$. The friction slope is defined in §4.5 and discussed in more detail in §3.8, but for now we simply note that Manning’s equation with $S_f$ is generally valid for any quasi-steady gradually-varying flow, which has made it a common model equation to relate $Q$ to frictional losses, even when modeling river physics with more complex unsteady-flow equations.

### 1.5.3 Saint-Venant’s equation

The approximations used in Manning’s equation are degraded when encountering time-varying flow rates or spatial gradients of the water surface slope, river geometry, or channel bed roughness. Alexandre de Saint-Venant derived more general differential conservation equations for mass and momentum [19], which may be written for a uniform density fluid as

$$\frac{\partial a}{\partial t} + \frac{\partial (au)}{\partial x} = 0$$

(1.11)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial \eta}{\partial x} - \frac{1}{R_h} f$$

(1.12)

where $g$ is gravity, $\eta$ is the local elevation of the fluid free surface, $f$ is a dimensional model of frictional losses, $x$ is an along-channel coordinate
that increases going downstream, and $R_h$ is the “hydraulic radius,” of eq. (1.8).

The Saint-Venant equations did not become a practical tool for river analysis until the advent of digital computers (e.g. [62]). However, computers were not a panacea: the Saint-Venant equations are difficult to solve for a natural river due to nonlinear relationships between $\eta$ and $a$, as well as $f$ being cast typically as a nonlinear function of $u$, $R_h$, and $n$. More than a half century of literature is replete with simplifications of the Saint-Venant equations, typically falling into the categories of “diffusive wave” or “kinematic wave” approximations [28]. These approximate models have some advantages in simplicity, numerical stability, and amenity to calibration. Although such models are likely to remain in practical engineering use for the near future, it seems likely that increases in computational power and data availability will render them obsolete. The Saint-Venant equations are the fundamental equations for river network analysis and are discussed in detail in §3.7.

1.6 Summary

In the same way that electric circuits have their idealized equations such as Ohm’s law, $I = V/R$, hydraulics has a suite of equations with varying levels of idealization. These were developed over the past 250 years by numerous scientists and engineers, but only some are commonly remembered in equation names: Chézy, Manning, and Saint-Venant. The key complicating factor for river hydraulics is the nonlinear relationship between the free-surface slope, i.e. $\partial \eta / \partial x$ in eq. (1.12), and the flow velocity. This relationship is further strained by the difficulty in accurately estimating the river geometry and bed composition, which affects flow resistance and has a nonlinear feedback into flow rates and free-surface gradients.
2

Electric Circuit Networks

2.1 Introduction

Analysis of river networks strongly resembles analysis of electric circuits, although the fundamental governing physical laws are different. In both cases the more general, albeit computationally more expensive, governing equations are simplified so that efficient analysis methods can be developed without sacrificing fidelity of the key processes and variables. More specifically, when water flows are constrained within the channels (§3.5) and the hydrostatic approximation is satisfied (§3.6), then the dynamic behavior can be described by systems of 1D Saint-Venant equations (§3.7) instead of the more general Navier-Stokes equations (§3.2). In electric circuit networks, Maxwell’s equations are approximated by Kirchhoff’s laws, as well as idealized lumped models of passive devices (resistors, capacitors and inductors) and nonlinear devices such as diodes and transistors. In this chapter, we briefly describe Maxwell’s equations (§2.2), which are further approximated by quasi-static assumptions and used to derive Kirchhoff’s laws (§2.3) as well as the idealized lumped models of some typical devices (§2.4). Simplified circuit network analysis is illustrated in §2.5, and the role of data in VLSI circuits is discussed (§2.6).
2.2 Maxwell’s equations

In electric circuits, the movement of information and energy is carried by electromagnetic waves. The general theory of electromagnetism is described by Maxwell’s equations, named after the 19th century physicist James Clark Maxwell, who proposed a unified theory of electromagnetism based on the work of earlier scientists including Carl Gauss, Michael Faraday, Joseph Henry, André-Marie Ampère and many others.

Maxwell’s equations can be written in different forms; the differential form is summarized below:

\[ \nabla \cdot \mathbf{D} = \rho \quad (2.1) \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.2) \]
\[ \nabla \cdot \mathbf{B} = 0 \quad (2.3) \]
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (2.4) \]

in which \( \mathbf{E} \) and \( \mathbf{D} \) are related to the electric field, while \( \mathbf{B} \) and \( \mathbf{H} \) are related to the magnetic field. \( \mathbf{E} \) is the vector field describing the electric field, \( \mathbf{D} = \epsilon \mathbf{E} \) is the vector field of electric displacement, or electric flux density. \( \mathbf{H} \) is the vector field describing the magnetic field, and \( \mathbf{B} = \mu \mathbf{H} \) is the magnetic flux density. \( \rho \) represents the charge density\(^1\) and \( \mathbf{J} \) is the current density.

Alternatively, Maxwell’s equations can be written in an integral form:

\[ \int_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dV \quad (2.5) \]
\[ \int_S \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \quad (2.6) \]
\[ \int_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad (2.7) \]
\[ \int_S \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{\partial}{\partial t} \int_S \mathbf{D} \cdot d\mathbf{S} \quad (2.8) \]

\(^1\)Note that in fluid mechanics, \( \rho \) is typically used to represent the fluid density; the use of the same symbol with different definitions is a challenge for any cross-disciplinary exercise.
2.2. Maxwell’s equations

Using either form, Maxwell’s equations have four components describing the properties of the electric field, the magnetic field, and how they are intertwined. The first equation, eq. (2.1) or eq. (2.5), is Gauss’s law for the electric force field: the flux of \( E \) flowing out of a closed surface equals the charge within the enclosed volume surrounded by the surface. The second equation, eq. (2.2) or eq. (2.6), is Faraday’s law: a varying magnetic field through a loop equals the line integral of electric field around the loop. The last two equations describe the property of the magnetic force field. Eq. (2.3) or eq. (2.7) is Gauss’s law for the magnetic field: the magnetic flux density flowing out of a close surface should equal zero. The latter is somewhat different from Gauss’s law of the electric field. The reason is that unlike the electric charge associated with the electric field, there is no such thing as magnetic charge, or at least it has not yet been observed. The last equation, eq. (2.4) or eq. (2.8), is Ampère’s law with Maxwell’s correction: a magnetic field around a loop can be induced either by the current flowing through, or the change of electric flux density through the loop. Due to space limitations, we cannot provide a detailed discussion of Maxwell’s equations, which is readily available in classic electromagnetism references such as [63, 73]. As a highly simplified description, Maxwell’s equations demonstrate that the electric and magnetic fields obey Gauss’s law, and an electric field can be induced by a time-varying magnetic field, and vice versa.

The “conservation of charge” provides a description of the electric field. The current is defined as the movement of (electric) charge, hence a charge conservation principle can be written in the differential form:

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \]  

(2.9)

or in the integral form:

\[ \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_V \rho \, dV \]  

(2.10)

In plain language, the principle of charge conservation states that the net current flowing into a closed surface should equal the build up of the charge over time inside the volume within the close surface. Note that the conservation of mass for fluid flow is identical to a conservation
of charge if \( \mathbf{J} \) is replaced with \( \rho \mathbf{u} \) where \( \rho \) is interpreted as the fluid density and \( \mathbf{u} \) is the velocity vector (§6.2). Neither mass nor charge can simply disappear in a conservative system.

### 2.3 Quasi-static approximation for circuit analysis

Although Maxwell’s equations provide a general theoretic framework for electromagnetism, directly applying these equations for practical problems can be computationally expensive. In particular for electric circuit analysis, the predominant model uses Kirchhoff’s laws, named after 19th century physicist Gustav Kirchhoff. There are two components to these laws: Kirchhoff’s current law (KCL) states that for any node in a circuit, the current flowing into the node should equal the current flowing out. In other words, the algebraic sum of currents flowing into any node in a circuit should equal zero:

\[
\sum_{k} I_k(t) = 0 \quad (2.11)
\]

where \( n \) is the number of edges connected to the particular node.

Another component is Kirchhoff’s voltage law (KVL), which deals with the loops in the circuit. It states that for any loop in a circuit, the sum of the voltage drops along the loop should equal zero. In other words, if we traverse the loop following its edges as shown in Fig. 2.1, when we come back to the starting node, we should not see any voltage difference:

\[
\sum_{k} V_k(t) = 0 \quad (2.12)
\]

where \( m \) is the number of edges in the particular loop. Note that KVL implies loops are a fundamental feature of electric circuits, in contrast to the inherent tree-branch structure that underlies river networks (§1.2). However, where acyclic loops occur in river networks (e.g. Fig. 1.4) the gradient water surface elevation (\( \eta \)), which represents the gravitational potential energy, can be integrated around a loop along a line

\[^2\text{Note the similarity to eq. (4.15) for conservation of flow at a river junction.}\]
s and must follow
\[ \int_s \frac{\partial \eta}{\partial s} ds = 0 \]

That is, any loop in a river must return to the same water surface elevation (the same gravity potential), which has little practical utility in a solution algorithm. However, for flow in pressurized pipe networks an equivalent to the KVL can be written for the pressure, which is useful in solving pipe network problems.

Figure 2.1: Circuit loop and lumped elements.

Although Kirchhoff derived the KCL and KVL independently before Maxwell’s work, we can derive Kirchhoff’s laws as a special case of Maxwell’s equations. This derivation requires the “quasi-static” assumption, which is predicated on the idea that analysis with static methods can be used while the associated electric and magnetic fields change rapidly\(^3\).  

\(^3\)This idea is similar to neglecting acoustic pressure waves in the hydrostatic approximation of fluid flow (§3.6).
Electric Circuit Networks

To illustrate this idea, consider a network of interconnected elements shown in Fig. 2.1. Here we also make an implicit assumption that the network consists of a collection of individual “lumped” elements, each with uniform characteristics. We further assume that the connecting wires between these lumped elements are ideal, i.e., they are perfect conductors with zero impedance. Considering an arbitrary loop \( L_p \) within the network, as shown in Fig. 2.1 if we apply Faraday’s law to this loop, we know that:

\[
\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} = -\int_S \frac{\partial}{\partial t} \mathbf{B} \cdot d\mathbf{S} \tag{2.13}
\]

We further introduce a “node” on each of these idealistic connecting wires, in particular at the intersection points of the wires, we can write the left-hand-site of eq. (2.13) as:

\[
\oint \mathbf{E} \cdot d\mathbf{l} = \int_{n_2}^{n_1} \mathbf{E} \cdot d\mathbf{l} + \int_{n_2}^{n_3} \mathbf{E} \cdot d\mathbf{l} + \cdots + \int_{n_m}^{n_1} \mathbf{E} \cdot d\mathbf{l} \tag{2.14}
\]

Next we introduce the concept of voltage, or potential difference, between two nodes along a path:

\[
V_{ab} = V_a - V_b = -\int_a^b \mathbf{E} \cdot d\mathbf{l} \tag{2.15}
\]

Therefore, the Faraday’s law for this arbitrary loop can be written as:

\[
V_{n_2n_1} + V_{n_3n_2} + \cdots + V_{n_mn_1} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{2.16}
\]

Finally, we invoke the “static” assumption, in particular the assumption that at the network level, the change of magnetic flux though this arbitrary loop \( L_p \) is zero. Hence \( \partial \mathbf{B}/\partial t = 0 \) and we arrive at Kirchhoff’s voltage law

\[
\sum_{k} V_k(t) = 0
\]

Because this voltage law holds for any arbitrary loop in the network of lumped elements, therefor often an electric component network is referred as an “electric circuits”\(^4\).

\(^{4}\)The term “circuit” came from Latin *circuitus*, which in turn came from *circumire* or *cireure*, *circum- + ire*, literally means “to go around”.
2.3. Quasi-static approximation for circuit analysis

Kirchhoff’s current law can be similarly derived from the principle of charge conservation. Again refer to the circuit shown in Fig. 2.1. For any node, we can draw a closed surface $S$ encircling it. Based on the charge conservation principle, we know that:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_V \rho \, dV = -\int_V \frac{\partial}{\partial t} \rho \, dV$$

Since all the wires are ideal with zero impedance, they will not hold charges. An alternative way of thinking is that if a particular wire can hold any charge, we can immediately introduce a lumped capacitor to model it, and connect it with two idealistic wires to the rest of circuit. In either case, the right-hand-side of the equation becomes zero. Again because the connecting wires have no impedance, the current $\mathbf{J}$ only flows inside the wires, therefore:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \sum_{k}^{n} \int_{S_k} \mathbf{J}_k \, dS_k$$

where $n$ is the number of idealistic wires connected to the node and $S_k$ is the cross section area of the wire. Because $\mathbf{J}_k$ is the current density and the integral of the current density with the cross section area is the total current within the wire, we arrive at Kirchhoff’s current law:

$$\sum_{k}^{n} I_k = 0$$

Note that two assumptions are required in the previous derivations: for any loop at circuit level, the change of the magnetic flux through the loop is zero (although the magnetic flux itself can be nonzero, e.g., the geomagnetic flux) and the circuit can be partitioned into multiple lumped models (§2.4) with ideal connectors. Generally speaking, these conditions imply the quasi-static assumption. The rule-of-thumb criteria when quasi-static assumption can be applied is that the dimensions of the elements under study are very small compared to the wavelength. For example, at the frequency of 1 MHz, the wavelength is about 300 meters (wavelength $\lambda = c/\nu$, where $c$ is the speed of light and $\nu$ is the frequency).

---

5This approach is identical to how the conservation of mass for fluid flow becomes the conservation of volume, eq. (3.1), for a uniform-density fluid.
frequency). If we are trying to analyze an amateur transistor radio, the quasi-static assumption will hold because the dimensions of the components are of millimeter scales. Another example is in contemporary VLSI electronic products, where the average dimension of a device is in the range of a few micrometers ($10^{-6}$ meter). The average operating frequency of the VLSI product is about 1 GHz ($10^9$ Hz), which translates to the wavelength of 0.3 meter. A typical modern VLSI design is less than 10 cm across and majority of the signal wires are only a few micrometers long; hence the quasi-static assumption holds for signal propagation inside a VLSI chip. However, the quasi-static assumption is invalid for signal propagation at print-circuit-board (PCB) level. Indeed, many PCB-level signal analyses cannot use Kirchhoff's laws, but instead apply more sophisticated electromagnetic wave analysis methods.

One of the biggest benefits of applying the quasi-static assumption (hence Kirchhoff's laws) is that the assumption neglects electromagnetic waves, which dramatically reduces computational complexity. Because electromagnetic waves propagate through both objects and empty space, a large three-dimensional region surrounding any circuit must be included in the simulation domain if the quasi-static assumption is not applied. The variety of materials used for microelectronics designs have different conductance properties for electric and magnetic fields, which introduces 3D spatial heterogeneity to the otherwise 1D circuit problem. The dimensions of conductors and insulators can be in the range of a few nanometers ($10^{-9}$ m), so a full electromagnetic simulation is very expensive. As a simple illustration, consider a VLSI design of $10 \times 10$ mm, using a Cartesian grid of 1 nanometer spacing the spatial discretization for a network in one dimension would require $10^7$ grid points. Extending the spatial discretization to full 3D space creates a computationally intractable problem.

Using Kirchhoff's laws, converting the quasi-static circuit analysis problem into a matrix format is straightforward. By applying modern numerical linear algebra and ordinary differential equation solving techniques, a complex circuit network can be efficiently analyzed. As modern VLSI designs can have billions of devices connected by hun-
dreds of billions of wires (each of which have to be modeled as multiple resistive-capacitive models), efficient simulation and optimization methods become critical to ensure the correctness and performance of the final product. For example, by applying Kirchhoff’s laws, on-chip power delivery networks with millions of resistors-capacitors can be analyzed in seconds [82].

The quasi-static approximation has similarities to both the use of the Chezy-Manning equation (§1.5.2) and the energy equation (§4.5) for river flow. These approaches are also used when the flow is static or changing slowly. The key difference between electric circuits and river channels with quasi-static approximations is that the former requires neglecting changes at the speed of light, while the latter requires neglecting changes at the speed of gravity waves, which leaves out important physics (§4.6) that are only captured by the full dynamic equations (§3.7).

2.4 Models of different lumped devices

2.4.1 Introduction

The quasi-static assumption implies that a circuit can be modeled by networks of multiple lumped models with ideal connectors. At the circuit level, we can use Kirchhoff’s laws to describe the overall circuit level behavior. At the device level, the lumped models are also called “Branch Constituent Relations” (BCR). In this section, we derive models of some of these devices. The first three are resistive (§2.4.2), inductive (§2.4.3), and capacitive (§2.4.4) devices, which are considered “passive” or linear. The last two are diodes and MOSFETs that are considered “active” or nonlinear (§2.4.5).

2.4.2 Resistive Device

A resistive path in the circuit is described by Ohm’s law:

\[
J = \sigma E
\]

which states that the current density in a conducting material is proportional to the electric field \(E\), where \(\sigma\) is the conductivity of the
material. Following the definition of potential difference, the voltage between two terminals $a$ and $b$ of an ideal conductor is:

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} \quad (2.18)$$

If we assume the conductivity of the material is location-dependent, $\sigma(l)$, where $l$ is a length along the conductor, along with spatially-varying cross-section area, $A(l)$, then the voltage between two terminals is:

$$V_{ab} = -\int_b^a \frac{Idl}{\sigma(l)A(l)} = -IR \quad (2.19)$$

where the resistance is defined as:

$$R = \int_b^a \frac{dl}{\sigma(l)A(l)} \quad (2.20)$$

When the conductor has uniform conductivity and cross section area $A$ and has the length of $\mathcal{L}$, the resistance model simplifies to the familiar form:

$$R = \frac{\mathcal{L}}{\sigma A} \quad (2.21)$$

Thus, Ohm’s law applied to a resistance element provides

$$I = \sigma A \frac{V}{\mathcal{L}}$$

Considering the Chezy-Manning equation for river flow ($\S 1.5.2$), if we define a river “conductivity” as $\sigma_r = n^{-1} R_h^{2/3}$ and the channel slope as the change in water surface elevation over the length, $S = \Delta \eta/\mathcal{L}_r$, where $\mathcal{L}_r$ is the river length, then

$$Q = \sigma_r A \sqrt{\frac{\Delta \eta}{\mathcal{L}_r}}$$

Thus, modeling a resistance element as quasi-static is directly analogous to the static Chezy-Manning model of a river element, although the latter is inherently nonlinear.

### 2.4.3 Inductive Device

An inductive device can store energy in the form of magnetic field. The simplest form of such a device is a coil made of conducting material
2.4. Models of different lumped devices

with one (or multiple) loops. A loop is defined as a closed circle following the single coil, with two terminals defined as node $a$ and $b$. Based on Faraday’s law, if we define a close surface around the inductor we can partition the integration into two components: one within the conductor and one outside of the conductor. Because we assume the coil has zero impedance, the integration inside the coil becomes identically zero:

$$\oint E \cdot dl = \int_{b(\text{inside})}^{a} E \cdot dl + \int_{b(\text{outside})}^{a} E \cdot dl = \int_{b(\text{outside})}^{a} E \cdot dl$$

(2.22)

Based on Faraday’s law, as well as the definition of voltage, we can write:

$$V_{ab} = -\int_{b(\text{outside})}^{a} E \cdot dl = \frac{d}{dt} \int_{S} B \cdot dS$$

(2.23)

By defining the “inductance” $L$ as the magnetic flux through the surface $S$ generated by unit current:

$$L = \frac{1}{I} \int B \cdot dS$$

(2.24)

We can rewrite the inductor model as:

$$V = \frac{d}{dt} (LI) = L \frac{dI}{dt}$$

(2.25)

Although we assume that the magnetic field is “static” at the circuit level, at the inductive component level this assumption is no longer applied, as in eq. (2.23).

This ideal inductor model assumes there is no energy storage related to the electric field and the resistance of the coil is zero. If the resistance of the coil cannot be ignored, a common practice is to further partition the coil model into two sub-components: a resistor in series with an ideal inductor. Of course, such a practice can be extended even further: we can construct more complicated models with multiple sub-components. We can also derive a model where one part of the circuit draws magnetic energy from another part of the circuit through two sets of coupled coils. Such a “mutual” inductor model is easily derived and can be found in any circuit analysis text book [60, 57].

An interesting observation is that from eq. (2.25), as the rate current change ($dI/dt$) increases the voltage also increases, which provides
a feedback damping the current increase. Therefore we can interpret an inductor as a “counter-weight,” which prevents the current from changing rapidly. Fluid mechanics has a similar relationship between water flow \((Q)\) and water surface elevation \((\eta)\) – the analogies to \(I\) and \(V\) discussed in §1.4. When the rate of water flow into a channel element \((dQ/dt)\) increases, the surface elevation tends to rise and therefore reduces the pressure gradient \((d\eta/dx)\), which provides a feedback damping the flow increase.

### 2.4.4 Capacitive Device

An ideal capacitor can be modeled as two parallel plates separated by non-conducting material, each plate connecting to the terminals with ideal connectors. Based on charge conservation principle, we can write:

\[
I = \frac{dQ}{dt}
\]  
(2.26)

where \(Q\) above is the standard notation for an electric charge (not a water flow rate!). We define the capacitance as the amount of charge held by a unit potential difference between the two plates: \(C = Q/V\). Hence:

\[
I = \frac{d}{dt} (CV) = C \frac{dV}{dt}
\]  
(2.27)

Or if we take the integral form:

\[
V = \frac{1}{C} \int I \, dt
\]  
(2.28)

A capacitor has no direct analogy in common river network features, although if the charge \(Q\) is mapped to a water volume and \(I\) to a water flow rate, then eq. [2.26] is a statement that the flow rate out of a reservoir is equal to the rate of change of the volume. It is possible that this idea might be useful in linking circuit modeling ideas to hydraulic structures that require complex representations (§4.11).

### 2.4.5 Models of Nonlinear Elements

The resistors, capacitors, and inductors discussed above are considered linear devices because their BCR can be expressed as linear equations
2.4. Models of different lumped devices

(if we consider the Laplacian to be a linear operator). Other devices have more complicated BCR and are quite often nonlinear. An example is a diode, which can be considered as an instance within a broader category of two-terminal nonlinear devices. Conceptually, a diode can be treated as a nonlinear resistor, whose resistance varies as the voltage changes. For example, the Shockley model of a p-n junction diode is:

\[ I = I_S \left( e^{\frac{V}{nV_T}} - 1 \right) \]  

(2.29)

where \( V \) is the voltage across the diode, \( V_T \) is a constant, \( n \) and \( I_S \) are device parameters, which are usually supplied by device manufacturers. Obviously this relationship is a more general form of the BCR for a regular resistor modeled by \( I = VR \). When a diode is present in a circuit, due to its nonlinear BCR, a nonlinear equation solving technique is required\(^6\).

Another widely used nonlinear device is the field effect transistors, in particular the metal-oxide-semiconductor field-effect transistor (MOSFET). Unlike a diode, a MOSFET has four terminals: gate (G), drain (D), source (S) and body (B), as shown in Fig. 2.2. The exact mechanism of a MOSFET device is beyond the scope of this paper; however, a simplified way to understand a MOSFET is that it behaves like a switch\(^7\). By controlling the voltage at the gate node (G), we can control the current flow between the drain (D) and source (S). The dominant equation of a MOSFET device is the current between drain (D) and source (S) nodes, \( I_{DS} \).

Generally speaking, the MOSFET BCR is a nonlinear function in the form of:

\[ I_{DS} = f(V_{GS}, V_{DS}, V_{BS}) \]  

(2.30)

The simplest example is the level-1 model:

\[ I_{DS} = \begin{cases} 
0 & V_{GS} < V_{th} \\
\beta[(V_{GS} - V_{th})V_{DS} - \frac{1}{2}V_{DS}^2] & 0 < V_{DS} < V_{GS} - V_{th} \\
\frac{\beta}{2}(V_{GS} - V_{th})^2 & 0 < V_{GS} - V_{th} < V_{DS}
\end{cases} \]  

(2.31)

\(^6\)Diodes are functionally similar to culverts or weirs in river networks (§4.11).

\(^7\)The equivalent to a MOSFET in hydraulics would be a gate arrangement to control flows in several canals; these typically are only seen in man-made systems for controlling irrigation and barge traffic.
in which threshold voltage $V_{th}$ and gain factor $\beta$ are device parameters. The first equation depicts the cut-off region, in which the transistor is completely off when the gate voltage $V_{GS}$ is below the threshold. As the gate voltage $V_{GS}$ is raised higher, the transistor enters linear region. However, if the driving voltage $V_{DS}$ is too high, the transistor enters the saturation region, in which the drain current $I_{DS}$ is no longer affected by the drain voltage $V_{DS}$, instead it is controlled by the gate voltage $V_{GS}$ only.

MOSFET device models are generally more complicated than the above example. Numerous second-order effects must be considered, including the sub-threshold leakage, the gate leakage, the body effect, and the short-channel effect. A compact device model can involve hundreds of coupled nonlinear equations \cite{69, 25}. Furthermore, beyond the current-voltage relationship a complete diode or MOSFET device model also involves complex capacitance-voltage relationships. More details can be found in device-modeling textbooks such as \cite{75} and \cite{72}.

From the point of view of river network analysis, MOSFET devices are interesting because they have a rich variety of nonlinear relationships. Thus, simulation methods for MOSFET devices might have direct applicability in handling the nonlinear coupling we illustrated in the analogies of §1.4.
2.5 Circuit network analysis based on Kirchhoff’s laws

One of the advantages of using Kirchhoff’s laws and BCR to analyze a circuit network is that the equations are readily formulated as a matrix problem that can be efficiently analyzed with modern numerical linear algebra techniques. The example in Fig. 2.3 illustrates this point. The circuit consists of four resistors and a capacitor, with one current source for excitation. Two widely used techniques to analyze the circuit are sparse tableau and modified nodal analysis (MNA). Here we use MNA formulation because we can construct the circuit model by simple inspection.

\[ I_{R_1} = \frac{V_1 - V_4}{R_1} \]

We can compute the current flowing through resistor \( R_2 \) similarly. Ac-
According to KCL, eq. (2.11), we have:

\[
\frac{V_1 - V_4}{R_1} + \frac{V_1 - V_2}{R_2} - I = 0 \tag{2.32}
\]

Similarly we can write the KCL of node 2 as:

\[
\frac{V_2 - V_1}{R_1} + \frac{V_2 - V_4}{R_3} + \frac{V_2 - V_3}{R_4} = 0 \tag{2.33}
\]

There is a capacitor at node 3. Following the BCR of a capacitor, we can write the KCL of node 3 as:

\[
\frac{V_3 - V_2}{R_4} + C \frac{d}{dt} (V_3 - V_4) = 0 \tag{2.34}
\]

Finally, the KCL of node 4 is

\[
\frac{V_4 - V_1}{R_1} + \frac{V_4 - V_2}{R_3} + C \frac{d}{dt} (V_4 - V_3) + I = 0 \tag{2.35}
\]

Thus, we have four unknowns (nodal voltages at nodes 1 to 4) as well as four KCL relations. We can write the KCL relations at four nodes in a matrix form:

\[
\begin{bmatrix}
\frac{1}{R_1} & \frac{1}{R_2} & -\frac{1}{R_3} & 0 & -\frac{1}{R_4} \\
-\frac{1}{R_1} & \frac{1}{R_2} & \frac{1}{R_3} & \frac{1}{R_4} & -\frac{1}{R_3} \\
0 & -\frac{1}{R_1} & \frac{1}{R_3} & \frac{1}{R_4} & 0 \\
-\frac{1}{R_1} & -\frac{1}{R_2} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
V_1(t) \\
V_2(t) \\
V_3(t) \\
V_4(t)
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & C & -C & 0 \\
0 & 0 & -C & C & 0
\end{bmatrix}
\begin{bmatrix}
V_1(t) \\
V_2(t) \\
V_3(t) \\
V_4(t)
\end{bmatrix}
= 
\begin{bmatrix}
I(t) \\
0 \\
0 \\
-I(t)
\end{bmatrix} \tag{2.36}
\]

The circuit described by eq. (2.36) is undetermined, e.g., there are infinite number of solutions which will satisfy the equations because Kirchhoff's laws only specify voltage (or potential) differences in the circuit. The problem can be uniquely determined by application of boundary conditions. In this case, we specify a voltage (usually zero) at a particular node, and this node becomes the reference point or ground node. This approach is similar to defining the water’s free surface in a river network, where the elevation (and hence potential energy) can
be specified by relative to any value, although usually mean sea level or a similar datum is used. For convenience, we choose the boundary condition in the example as $V_4 = 0$. Thus we remove one unknown and one KCL from the system of equations in eq. (2.36), which becomes:

$$
\begin{bmatrix}
\frac{1}{R_1} + \frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} & 0 \\
-\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4} & \frac{1}{R_4} \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
V_1(t) \\
V_2(t) \\
V_3(t)
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
I(t) \\
0 \\
0
\end{bmatrix}
$$

(2.37)

The above is a system of differential-algebraic equations (DAE) requiring $I(t)$ as a temporal boundary condition. For illustrative purposes, we assume that the input excitation of the current source is a Heaviside step function of the form

$$
I(t) = \begin{cases}
0 & t < 0 \\
1 & t \geq 0
\end{cases}
$$

First let’s consider the DC (or steady-state) solution, which is equivalent to the solution when $t \to \infty$. When the circuit has been given sufficient large time to settle, the capacitor is fully charged and can be treated as an open circuit. An alternative viewpoint is that the nodal voltages are constant at steady state, therefore $dV/dt = 0$ for any node. Thus, for the DAE of eq. (2.37) the second matrix on the left-hand-side evaluates to zero. For the simple case where all the resistors are equal with same resistance of 1Ω, the DC solution of the circuit is determined by solving the following matrix equation:

$$
\begin{bmatrix}
2 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
$$

(2.38)

It can be easily shown that the DC solution of the circuit is $V_1 = 0.667V$ and $V_2 = V_3 = 0.333V$.

---

*Here we assume that the resistance, current and voltage are all in SI units shown in Table 1.2.*
To solve the unsteady (or transient) solution of this circuit, we require a time-marching integration method for the DAE and a set of initial conditions for all nodes. Since the excitation is a step function from a null state, the appropriate initial condition of all nodes is \( V_1(0) = V_2(0) = V_3(0) = 0 \). There are many suitable time-marching methods to solve this DAE. For illustrative simplicity, the one-step backward Euler’s method can be used, wherein the derivative of any function \( f' \) is approximated from time step \( k - 1 \) to \( k \) as

\[
  f' \approx \frac{1}{\Delta t} \{ f(k) - f(k - 1) \}
\]

which implies

\[
\begin{bmatrix}
  V'_1(k) \\
  V'_2(k) \\
  V'_3(k)
\end{bmatrix} \approx \frac{1}{\Delta t} \begin{bmatrix}
  V_1(k) \\
  V_2(k) \\
  V_3(k)
\end{bmatrix} - \begin{bmatrix}
  V_1(k - 1) \\
  V_2(k - 1) \\
  V_3(k - 1)
\end{bmatrix}
\]

(2.39)

At time point \( k \), the solutions at \( k - 1 \) are already known, so we can re-arrange the \( k - 1 \) terms to the right-hand-side of the equation. Thus, to solve for the solution at time point \( k \), we need to solve the matrix equation:

\[
\begin{bmatrix}
  \frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} & -\frac{1}{\mathcal{R}_2} & 0 \\
  -\frac{1}{\mathcal{R}_2} & \frac{1}{\mathcal{R}_2} + \frac{1}{\mathcal{C}} & -\frac{1}{\mathcal{R}_4} \\
  0 & -\frac{1}{\mathcal{R}_4} & \frac{1}{\mathcal{R}_4} + \frac{\mathcal{C}}{\Delta t}
\end{bmatrix}
\begin{bmatrix}
  V_1(k) \\
  V_2(k) \\
  V_3(k)
\end{bmatrix} = \begin{bmatrix}
  I(k) \\
  0 \\
  0
\end{bmatrix} + \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & \frac{\mathcal{C}}{\Delta t}
\end{bmatrix}
\begin{bmatrix}
  V_1(k - 1) \\
  V_2(k - 1) \\
  V_3(k - 1)
\end{bmatrix}
\]

(2.40)

For practical applications, the time step is usually adjusted based on the estimation of local truncation error at each time point. Higher order integration methods such as Backward Differentiation Formula methods (BDF) are commonly used with this type of stiff DAE system.

As a general observation, the matrix representing the circuit is generally sparse. Because each column corresponds to a node in the circuit, and almost all nodes have only a handful of adjacent nodes, the number of non-zero elements in each row is small. The only exception
is the reference node (or ground), which could be connected to many nodes. However, since we artificially set the ground voltage to zero and exclude it from the unknown vector, we effectively remove the corresponding matrix row and column associated with the ground.

2.6 The role of data in electric circuits

To simulate the electric circuits on a VLSI chip, it is necessary to compute the model parameters (e.g., resistance, capacitance, inductance, as well as parameters of the nonlinear diodes and transistors). Due to the minuscule scale of the components (in the order of $10^{-9}$ m) and large quantity (in the order of $10^9$ of transistors, much more if resistive and capacitive components are included), it is impossible to measure the parameters of individual components. Instead, the blueprint of the design (often referred as the “layout”) and pre-characterized instances (often referred as “libraries”) are used to compute the model parameters automatically, a process often referred as “extraction”. In contemporary semiconductor manufacturing technology, inevitably there are uncertainties in the model parameters [47], which can affect the overall performance of the final product. A comprehensive review on these topics is beyond the scope of this manuscript. Here we use a simple example on the uncertainties in resistance extraction to illustrate the issue.

Conceptually a contemporary VLSI design can be visualized as a sea of switches (transistors) connected by metal wires. The conducting wires carry signals between, and power to, the switches. The wires are usually made of various conducting materials, such as doped polycrystalline silicon, aluminum, copper, and sometimes tungsten or platinum. Fig. 2.4 demonstrates an example of a few transistors and connecting wires in a 3D view.

To compute the resistance of a connecting wire, we can use the dimensions in the layout and the resistivity model shown in Eq. (2.21). However, this simple model breaks down when the wires makes a 90 degree turn or at a T-shape junction because the simple model of Eq. (2.21) is only an 1D approximation of the 3D physics. The prob-
A problem could be formulated and solved as a 3D Laplace equation; however, this approach is prohibitively expensive due to the large number of wires on a microchip. To retain a 1D network problem that can be rapidly solved, the 3D features are represented by sub-models that quantify their net effect (e.g., effective resistance) on the circuit [41]. This approach is practical because semiconductor manufacturing processes impose restrictions on the layout. For example, there are limited choices for metal thickness and width, all wires have to be either horizontal or vertical, and no curved wires are allowed. Such restrictions limit the possible permutations of bends and junctions, allowing the Laplace equation for possible configurations to be pre-solved and stored in a library. This approximation approach is acceptable because we do not need to know the 3D current distribution inside a wire, our only concern is for the net circuit effect.
2.6. The role of data in electric circuits

In the past decade, the dimensions of the semiconductor products have been reduced to the level that atomic scale effects become significant \[47\]. A comprehensive discussion on different atomic effects and their manifestations is beyond the scope of this manuscript. For wire resistance, the important point is that the dimensions of the manufactured parts can significantly deviate from the blueprint (or layout) specifications. Fig. 2.5 shows a simulation of manufactured metal wire illustrating the wire complexity at small scales. To add further difficulty, the deviation magnitudes (often referred as “process variations” or “manufacturing variations”) have complicated spatial and temporal behavior. In other words, for the same type of wire, the deviation at one corner of a single VLSI chip can be different from deviation at the other corner. Because the VLSI chips are manufactured in batches, the variation can also exist across different chips, i.e. a particular location on one chip may have a different deviation than the identical location on an otherwise identical chip manufactured at a different time. A common design approach is to provide a range of parameters (e.g., typical, worst case, and best case values). This problem is dealt with in the literature at both modeling and simulation level, e.g. \[53\] \[83\] \[14\].

Figure 2.5: Simulated results of the manufactured shapes on semiconductor shapes. URL: http://www.mikroelektronik.fraunhofer.de

The issues associated with uncertainty in the geometric data (and
hence effects on resistance) for an electric circuit are quite similar to those for river networks, particularly for the river cross-sectional geometry (§4.8, §4.9). The problems associated with modeling 3D wire features as 1D for computational efficiency is precisely the problem encountered in deriving the 1D Saint-Venant equations from the 3D Navier-Stokes equations (§3.2).

2.7 Summary

Electric circuit networks are governed by Maxwell’s equations, which are reduced to Kirchhoff’s laws under a quasi-static approximation. The circuit model becomes a simplified system of lumped elements governed by Branch Constituent Relations (BCR) with ideal connections to nodes. The elements can be linear (resistive, inductive, capacitive) or more general nonlinear BCR for diodes, transistors or other semiconductor devices. The circuit is modeled as a set of Discrete Algebraic Equations (DAE), as compared to the Partial Differential Equations (PDE) that govern flow physics in a river (Chapter 3).
3

Governing Equations for River Flow

3.1 Introduction

Before we can examine river networks, it is useful to consider the fundamental equations that govern flow in a single channel. Although river flow is inherently a 3D phenomena, large-scale models have traditionally used 1D models derived from first principles. The meanings and effects of the various terms and simplifications in the 1D models are best understood by examining the 3D equations, which become our departure point in this section. However, readers interested specifically the standard 1D governing equations for river networks can skip directly to §3.7 and the discussion of the friction slope in §3.8, then continue to Chapter 4 for an introduction to river networks.

The discussions below of Reynolds-averaging (§3.3), channel curvature (§3.4), cross-section averaging (§3.5), and the hydrostatic approximation (§3.6) are topics that are omitted from much of the river modeling literature; these are foundational ideas that are simply accepted as known and mentioned (if at all) as standard approximations. We present derivations and discussions as a reference for both electrical engineers and hydraulic engineers, since these derivations cannot be readily found in a condensed form in any single source, except perhaps
Governing Equations for River Flow

[17], which is out of print. It is our hope that cross-disciplinary review of these standard ideas might lead to new insights and approaches.

3.2 Three-dimensional equations

At the most fundamental level, the governing equations for fluid flow are the conservation of mass (or continuity) and the conservation of momentum; in 3D the latter are commonly known as the Navier-Stokes equations. We consider the simplified case of a uniform-density incompressible liquid flow in Cartesian space, where the conservation of mass becomes a divergence-free condition on the velocity, sometimes known as the conservation of volume:

$$\frac{\partial u_j}{\partial x_j} = 0$$

(3.1)

where $j = \{1, 2, 3\}$, $u_j$ is the velocity in the $x_j$ coordinate direction, and the Einstein summation convention is applied for repeated subscripts. Note that incompressible conservation of mass has an inherent unphysicality; if we imagine a length of pipe governed by eq. (3.1), this divergence-free condition requires that any fluid pushed into the upstream end of the pipe must immediately cause an equal volume to be pushed out of the downstream end – i.e. the information passes downstream without any delay. This incompressibility approximation neglects pressure waves traveling at the speed of sound, which physically cause temporary local increases in fluid density and control the delay between an input impulse and its outflow effect. The unphysicality of the incompressibility approximation is acceptable for river channels because free-surface gravity waves (§4.6) provide the delay behavior that dominates water compressibility effects in open-channel flow. That is, an impulse introduced into the upstream end of an open channel does not result in an immediate impulse at the channel outlet – even with the incompressibility approximation. The delay between input impulse and outflow effect is controlled by the speed of gravity waves on the water’s surface, which are slower but carry more energy than acoustic waves associated with pressure compression. Note that the incompressibility approximation does not neglect pressure in its entirety (see §3.6),
just the pressure gradients associated with high-speed acoustic waves and the slight compressibility of a liquid. An analogy can be drawn between neglecting delays associated with speed of sound in water and neglecting delays associated with the speed of light in electric circuits (§2.3).

Conservation of momentum per unit mass (i.e. velocity) can be written as the net change in momentum balanced by the net forces on a differential volume:

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + g_i
\] (3.2)

where \(\nu\) is the fluid kinematic viscosity, \(\rho\) is the fluid density, \(p\) is the pressure, \(g_i\) is the force of gravity. The principal assumption of eq. (3.2) is the shear stresses in the fluid are governed by a Newtonian viscous model – the stress is proportional to strain with viscosity as the proportionality coefficient – which is reasonable for water flow [3]. The 3D equations are usually invoked with the \(x_3\) axis aligned with the vertical direction.

Neglecting wind stresses on the water surface, erosion/aggradation of the solid boundary, as well as surface tension effects at the free surface (\(\eta\)) and bottom (\(b\)), the boundary conditions for velocity, pressure, and the free surface elevation are:

\[
(u_i)_b = 0 \quad (3.3)
\]
\[
\left( \frac{\partial u_1}{\partial x_3} \right)_{\eta} = 0 \quad (3.4)
\]
\[
\left( \frac{\partial u_2}{\partial x_3} \right)_{\eta} = 0 \quad (3.5)
\]
\[
\frac{\partial \eta}{\partial t} + u_1 \frac{\partial \eta}{\partial x_1} + u_2 \frac{\partial \eta}{\partial x_2} + u_3 = 0 \quad (3.6)
\]
\[
p_{\eta} = 0 \quad (3.7)
\]
\[
\left( \frac{\partial p}{\partial x_3} \right)_b = 0 \quad (3.8)
\]

where the variable \(\eta\) is the elevation of the free surface above a horizontal baseline \(x_3 = 0\).
Direct numerical solution of the 3D governing equations requires model grid scales on the order of the Kolmogorov length scale for turbulence [74], which is less than \( O(10^{-3} \, \text{m}) \) for typical river conditions. Solution at such fine scales is computationally impractical for any sufficiently large river network with present computers. Furthermore, our inability to accurately map the bottom boundary at such fine spatial scales (especially given the continual temporal evolution of the boundary) makes direct solution likely to remain impractical, regardless of computational power [37]. That is, how can one precisely solve the spatio-temporal evolution for the exact governing equations in a river channel if the boundary conditions are only approximately known? Practical river models must solve a set of approximate governing equations for smoother, large-scale flow behavior with empirical relationships for smaller-scale flow features [17].

3.3 Reynolds-averaged Navier-Stokes equations

A time-averaged set of equations is commonly used in river modeling. The averaging time scale, \( \Delta t \), is typically taken as the time step used for an unsteady, time-marching solution. The mean value of a variable over the time scale is defined (using the velocity for example) as

\[
U_i \equiv \frac{1}{\Delta t} \int_{\Delta t} u_i dt \quad (3.9)
\]

and the fluctuation from the mean is

\[
u'_i \equiv u_i - U_i \quad (3.10)
\]

The decomposition into time-averaged and fluctuating parts (i.e. \( u_i = U_i + u'_i \)) is substituted into eqs. (3.1) and (3.2). The entire equations are time-averaged, providing

\[
\frac{\partial U_j}{\partial x_j} = 0 ; \ i, j = \{1, 2, 3\} . \quad (3.11)
\]

\[
\frac{\partial U_i}{\partial t} + \frac{\partial}{\partial x_j} (U_i U_j) = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - g_i - \frac{\partial}{\partial x_j} u'_i u'_j \quad (3.12)
\]
where the overbar indicates a time average of a product and all capital letters indicate the time-average of the corresponding lower-case variable. These equations are commonly known as the Reynolds-Averaged Navier-Stokes (RANS) equations \[61\]. Comparing eq. (3.12) with (3.2), the effect of time-averaging on the equation structure is the appearance of “Reynolds stress” gradients as the last term of eq. (3.12). The Reynolds-stress term represents the nonlinear contribution to energy losses caused by unsteady 3D turbulent motions redistributing momentum over time scales smaller than $\Delta t$. In common turbulent flows in the natural environment, the effects of small-scale advection dominate the viscous forces associated with time-averaged velocities, that is

$$\frac{\partial}{\partial x_j} u_i' u_j' \gg \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j}$$

so the viscous term is generally neglected in eq. (3.12). When viscous forces actually appear in a set of governing equations, they are typically understood to be using an “eddy viscosity”, $\nu_e$, to represent a model of the Reynolds stresses, i.e.

$$\nu_e \frac{\partial^2 U_i}{\partial x_j \partial x_j} \approx \frac{\partial}{\partial x_j} u_i' u_j'$$

Unfortunately, a single $\nu_e$ cannot adequately account for the tensor structure implied by the Reynolds stresses.

Extensive work on turbulence modeling has been focused on finding ways to close the set of equations caused by the introduction of the six $u_i' u_j'$ terms. For our purposes, it is convenient to represent these terms by a generic turbulence model $G_i$, that serves to diffuse momentum and dissipate energy. For convenience, we will retain the original nomenclature of $u$ and $p$ to represent the Reynolds-averaged velocity and pressure in the “conservative” equation form:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} (u_i u_j) = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - G_i - g_i$$

(3.15)

$$\frac{\partial u_j}{\partial x_i} = 0$$

(3.16)
where

\[ G_i \equiv \frac{\partial}{\partial x_j} u_i u_j \]  \hspace{1cm} (3.17)

Note the conservative form of the Navier-Stokes equations arises by taking advantage of the incompressibility assumption inherent in eq. (3.16) so that \( u_j \) can pass through the spatial gradient in the nonlinear term of eq. (3.2).

### 3.4 Channel curvature

The Cartesian coordinate system inherent in eqs. (3.15) and (3.16) is inconvenient for a sinuous river as the velocity in the streamwise direction is typically much larger than the cross-channel velocity. Because river channels change direction, the principal flow axis will alternately align with or be at an angle to any fixed Cartesian axes. It is useful to reformulate the equations into a coordinate system using a smoothed channel centerline as a sinuous \( x \) axis with the \( y \) coordinate defined everywhere locally-perpendicular to the channel centerline. A channel-conforming coordinate system is well-defined as long as the radius of curvature of the channel centerline \( (r_c) \) is greater than the channel width. Of course, a channel-forming coordinate system no longer has the same spatial properties as a Cartesian system, so the Navier-Stokes equations must be either derived in generalized curvilinear coordinates, or transformed using the chain rule of partial differentiation [3].

The Navier-Stokes equations can be rigorously transformed into a river coordinate system and then simplified with a perturbation approximation [38]. The result for eq. (3.15) can be presented as equations in three curvilinear coordinates, which for simplicity in exposition we define as \( x, y \) and \( z \) directions:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = - \left( 1 - 2\epsilon \right) \frac{1}{\rho} \frac{\partial p}{\partial x} - G_1 - \frac{2}{r_c} uv \]  \hspace{1cm} (3.18)

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial y} - G_2 + \frac{1}{r_c} u^2 \]  \hspace{1cm} (3.19)

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} - G_3 - g \]  \hspace{1cm} (3.20)
3.5. Cross-section integration

where \( u, v, \) and \( w \) are the velocity components along the curvilinear \( x, y, \) and vertical \( z \) axes, \( G_1, G_2 \) and \( G_3 \) are coordinate-transformed RANS turbulence models, and \( \epsilon = \epsilon(x, y) \) is a perturbation parameter defined from the local radius of curvature \( r(x, y) \) and the radius of curvature for the channel centerline at \( r_c = r(x, 0) \):

\[
\epsilon(x, y) = \frac{r(x, y) - r_c(x, 0)}{r_c(x, 0)}
\]  

(3.21)

For continuity the equivalent of eq. (3.16) is

\[
\frac{\partial}{\partial x} ((1 + \epsilon) u) + \frac{\partial}{\partial y} ((1 + \epsilon) v) + \frac{\partial w}{\partial z} = 0
\]  

(3.22)

Thus eqs. (3.15) and (3.16) are an approximation of eqs. (3.18) – (3.20) and (3.22) in channel-conforming curvilinear coordinates neglecting terms of \( O(\epsilon) \), and \( O(r^{-1}_c) \), which are small for any smoothly sinusous river. The effects of these small curvilinear perturbation terms is to induce a helical flow in the channel bend that affects frictional losses. This flow is inherently nonlinear and scales on the channel depth, such that it is difficult to adequately parameterize in a 1D model. Although the large-scale curvature has been neglected in 1D hydraulic river models, it is hoped that the presentation herein might inspire some cross-disciplinary ideas towards its future inclusion, perhaps using some of the ideas in §2.6.

3.5 Cross-section integration

3.5.1 Momentum

Neglecting the effects of channel curvature discussed in §3.4, integration of eq. (3.15) over a channel cross-section of area \( a \) results in

\[
\frac{\partial a \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_1} (\beta a \bar{u}_1 \bar{u}_i) + \frac{1}{\rho} \int_a \frac{\partial p}{\partial x_i} da + \int_a G_i da + M_i + F_i = 0
\]  

(3.23)

where \( \bar{u}_i \) are the cross-section averaged velocities and \( i = \{1, 2, 3\} \) without application of the Einstein summation convention. This cross-section integration introduces several new terms in the conservation of momentum. The \( G_i \) term is still the turbulence model of nonlinear, small-time-scale Reynolds stresses, but now is integrated over the
The factor $\beta_i$ represents the non-uniformity of the velocity field over the cross-section, defined as

$$\beta_i \equiv \frac{a \bar{u_i} \bar{u_1} + \int_a \bar{u_i} \bar{u_1} \, da}{a \bar{u_i} \bar{u_1}}$$  \hspace{1cm} (3.24)$$

where $\bar{u}$ is the difference between the velocity at any point and the cross-section average:

$$\bar{u}_i \equiv u_i - \bar{u}_i$$  \hspace{1cm} (3.25)$$

Thus, $\beta_i$ represents effects of flow structures that have longer time-scales than turbulence (i.e. those represented by $G_i$), which cause cross-channel non-uniformity in the velocity field. The general presumption is that the non-uniformity is small compared to the downstream velocity $u_1$, i.e.

$$\bar{u}_i \bar{u}_1 \gg \int_a \bar{u_i} \bar{u_1} \, da$$  \hspace{1cm} (3.26)$$

hence $\beta \sim 1$ is an accepted approximation.

$M_i$ accounts for movement of the river bottom, defined by elevation $b(x, y)$ that is integrated from the left bank of the cross-section ($y_L$) to the right bank ($y_R$)

$$M_i \equiv \int_{y_L}^{y_R} (u_i) b \left[ \frac{db}{dt} + (u_1) b \right] dy$$  \hspace{1cm} (3.27)$$

where $(u_i) b$ is a velocity of the moving boundary in the $i$ coordinate direction. $M_i$ is typically neglected for slow erosion and aggradation unless models are being used to directly analyze these processes [12,68].

$F_i$ represents the momentum flux through the boundaries, typically due to confluence of tributaries or groundwater springs.

$$F_i \equiv - \int_{y_L}^{y_R} (u_i u_3) b \, dy + \int_{\tilde{b}}^{\bar{\eta}} \left[ (u_i u_2)_{yR(z)} - (u_i u_2)_{yL(z)} \right] \, dz$$  \hspace{1cm} (3.28)$$

where $\tilde{b}$ is the lowest point in the channel and $\bar{\eta}$ is the average free surface elevation. The first term on the RHS of eq. (3.28) represents momentum effects of vertical inflows (e.g. groundwater) while the second term represents net side inflows from river junctions. Groundwater inflows are generally expected to have small horizontal velocities, i.e. $u_1 \sim u_2 \sim 0$, so the first term provides negligible contributions to
horizontal momentum. In contrast, river junctions may provide significant horizontal momentum contributions through the second term, but these contributions are difficult to estimate due to the complex hydrodynamic and geomorphic structure at a confluence of rivers [20]. Furthermore, the same complexities that make $F_i$ difficult to estimate affect $G_i$, which typically provides increasing energy losses at a junction. As a result, changes in $F_i$ at junctions are often considered implicitly offset by $G_i$, so $F_i$ is usually neglected and $G_i$ retains the same structure at a junction that it does in a simple channel. Note that models which retain $F_i$ but fail to model changes in $G_i$ are inconsistent and unlikely to provide a more accurate simulation. Thus, the simplified cross-section integrated momentum equation is

$$\frac{\partial a \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_1} (\beta_i a \bar{u}_i \bar{u}_i) + \frac{1}{\rho} \int_a \frac{\partial p}{\partial x_i} \, da + \int_a G_i \, da = 0$$

(3.29)

which implies three equations ($i = 1, 2, 3$) that are coupled through the along-channel velocity ($\bar{u}_1$) and pressure ($p$).

### 3.5.2 Continuity

Cross-section integration of eq. (3.16) using Leibnitz rule and the kinematic boundary condition for a free surface, eq. (3.6), results in

$$\frac{\partial}{\partial t} \int_a \eta \, da + \frac{\partial}{\partial x_1} (a \bar{u}_1) - q_{yE} - q_{bE} - B = 0$$

(3.30)

where $B$ is the effect of bottom motion and $q_{yE}$ and $q_{bE}$ are the net inflow rates per unit length introduced through transverse flows (tributaries) and bottom flows (groundwater), respectively. Note that unlike the momentum effects of the $F_i$ terms in eq. (3.23), these inflow volume effects cannot be neglected. However, the $B$ terms in eq. (3.30) are typically neglected unless a model is specifically designed for sediment transport and bed erosion/aggradation. With the bottom motion ($B$) neglected, the rate of change of cross-sectional flow area depends only on the free surface motion, i.e.

$$\frac{\partial a}{\partial t} = \frac{\partial}{\partial t} \int_a \eta \, da$$

(3.31)
So continuity is typically presented as

\[ \frac{\partial a}{\partial t} + \frac{\partial}{\partial x_1} (a\bar{u}_1) = q_{y\ell} + q_{b\ell} \quad (3.32) \]

Note that presentations in the literature often leave out the \( q_{y\ell} \) and \( q_{b\ell} \) fluxes, providing a RHS of zero for simplicity.

### 3.6 Hydrostatic approximation

Eqs. (3.29) and (3.32) form a set of four equations with five unknowns, \( a, \bar{u}_1, \bar{u}_2, \bar{u}_3 \) and \( p \), along models required for \( \beta_i \) and \( G_i \). Further approximations are required to obtain a solvable equation set. We note that velocities in cross-channel \( (u_2) \) and vertical \( (u_3) \) directions are only directly coupled to the average streamwise velocity \( (u_1) \) equation through the pressure term. To address this coupling, we decompose the pressure into hydrostatic and non-hydrostatic parts, where the hydrostatic term represents the weight of the fluid and the non-hydrostatic term represents normal forces associated with accelerating or decelerating a fluid. The hydrostatic pressure is the force experienced by a deep-sea diver caused by the weight of the water and atmosphere above, whereas the non-hydrostatic pressure is the force experienced by a protester held back by a water cannon or fire hose. The pressure decomposition is

\[ p = \rho g (\eta - z) + p_{nh} \quad (3.33) \]

where \( p_{nh} \) is the non-hydrostatic portion of pressure and \( z \) is local elevation measured vertically from a baseline \( z = 0 \). The hydrostatic approximation (also known as the shallow-water approximation) allows us to neglect \( p_{nh} \) so that \( p \approx \rho g (\eta - z) \). This approximation is valid where horizontal length scales are large compared to vertical length scales, which is generally true for most rivers – especially when focusing on the streamwise \( (u_1) \) flow. Invoking the hydrostatic approximation, the streamwise momentum, from eq. (3.29), becomes

\[ \frac{\partial \bar{u}_1}{\partial t} + \frac{\partial}{\partial x_1} (\beta_1 a\bar{u}_1 \bar{u}_1) + g a \frac{\partial \bar{\eta}}{\partial x_1} + \int_a G_1 da = 0 \quad (3.34) \]

where \( \bar{\eta} \) indicates the cross-section averaged free surface elevation. Note that this equation is no longer coupled to the cross-stream or vertical
3.7 Traditional 1D Saint-Venant equations

The Saint-Venant equations have a number of traditional forms, depending on the choice of variables for solution and the approach to treating geometry [17]. The general approach outlined in §3.5 and §3.6 can be written as

\[
\frac{\partial a}{\partial t} + \frac{\partial}{\partial x} (au) = q_y + q_b \tag{3.38}
\]

\[
\eta = f(a) \tag{3.39}
\]

where the terms in eq. (3.37) can be compared to the terms in eq. (3.2). The notation in eqs. (3.37) - (3.39) is simplified from §3.5 and §3.6 because all terms are understood to be cross-section averages and the

momentum equations (except indirectly through the turbulence model term \(G_1\)). Furthermore, cross-stream and vertical velocities are small compared to the streamwise velocities, i.e.

\[
\bar{u}_1 \gg |\bar{u}_2| \quad ; \quad \bar{u}_1 \gg |\bar{u}_3| \tag{3.35}
\]

so we are free to drop the \(\bar{u}_2\) and \(\bar{u}_3\) equations implied by eq. (3.29), leaving us with the coupled eqs. (3.32) and (3.34) with unknowns \(a, \bar{u}_1\), and \(\bar{\eta}\). Assuming we know the cross-section geometry of the river, we can define a functional relationship of

\[
a = f(\bar{\eta}) \tag{3.36}
\]

so the equation set is closed.

Note that one of the difficulties in applying the effects of channel curvature (§3.4) in a 1D model is that the effects scale on the cross-channel and vertical velocities that inherently have non-hydrostatic behavior in a river channel. Thus, dropping the \(\bar{u}_2\) and \(\bar{u}_3\) equations for the hydrostatic approximation is arguably inconsistent with including channel curvature.
only coordinate direction is the streamwise ($x$) direction. In eq. (3.37) a “friction slope” $S_f$ is introduced as an empirical model to represent the cross-section average of the turbulence terms. The name “friction slope” arises as this term can be interpreted as the slope of a normalized energy gradient in the streamwise direction, thus it represents the frictional energy losses along the channel that arise as viscous effects are distributed through turbulent motion (see §4.5, §3.8).

An alternative Saint-Venant formulation uses the river volumetric flow rate, $Q = au$, as a variable, resulting in

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \beta \frac{Q^2}{a} \right) = -g a \frac{\partial \eta}{\partial x} - g a S_f$$  \hspace{1cm} (3.40)

$$\frac{\partial a}{\partial t} + \frac{\partial Q}{\partial x} = q_y + q_b$$  \hspace{1cm} (3.41)

which is also closed with eq. (3.39).

In either eq. (3.37) or (3.40), we can change the water surface elevation ($\eta$) into a depth variable ($h$), by defining

$$h \equiv \eta - z_b$$  \hspace{1cm} (3.42)

where $z_b$ is the vertical elevation of the channel bottom (typically taken as the lowest point in the channel at a cross-section). The bottom slope ($S_0$) is defined as

$$S_0 \equiv \frac{\partial z_b}{\partial x}$$  \hspace{1cm} (3.43)

so that eq. (3.37) is rewritten as

$$\frac{\partial au}{\partial t} + \frac{\partial}{\partial x} \left( \beta au^2 \right) + g a \frac{\partial h}{\partial x} = g a (S_0 - S_f)$$  \hspace{1cm} (3.44)

and eq. (3.40) becomes

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \beta \frac{Q^2}{a} \right) + g a \frac{\partial h}{\partial x} = g a (S_0 - S_f)$$  \hspace{1cm} (3.45)

These are closed with eqs. (3.38) or (3.41) along with the geometric relationship

$$h = f(a)$$  \hspace{1cm} (3.46)
3.7. Traditional 1D Saint-Venant equations

As a further modification, if the depth \((h)\) is desired as a dependent variable in continuity and momentum, we can note that for infinitesimal changes in the depth,

\[
\frac{\partial a}{\partial t} \approx \sigma(h) \frac{\partial h}{\partial t}
\]

where \(\sigma(h)\) is the channel cross-section breadth at the free surface. Thus eq. (3.41) becomes

\[
\sigma(h) \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = q_y + q_b
\]

and

\[
a = f(h)
\]

is used to close the equation set.

A slightly different set of Saint-Venant equations can be derived by integration over short streamwise section of a river reach where the channel curvature is neglected. This approach, as comprehensively derived in [17] and used by many authors, e.g. [4], introduces geometric terms \(I_1\) and \(I_2\) so that 1D momentum is written as

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{a} + gI_1 \right) = ga (S_0 - S_f) + gI_2
\]

which is closed by eq. (3.38) and where geometric auxiliary relationships are defined as

\[
I_1 = \int_b^\eta (\eta - z) \sigma \, dz
\]

\[
I_2 = \int_b^\eta (\eta - z) \frac{\partial \sigma}{\partial x} \, dz
\]

In the above, \(\eta\) and \(b\) are vertical elevations from a consistent baseline representing the average surface elevation and the minimum bottom elevation, respectively; \(\sigma = \sigma(z)\) is the channel width as a function of elevation.

Thus, the Saint-Venant equations may be formulated with different possible variable sets: \(\{Q, h\}\), \(\{Q, a\}\), \(\{Q, \eta\}\), \(\{u, h\}\), \(\{u, a\}\), or \(\{u, \eta\}\). Independent geometric relationships for each river cross-section are required for relationships of \(h(a)\), \(\eta(a)\), \(a(h)\), \(a(\eta)\) or \(I_1(\eta, \sigma)\) and \(I_2(\eta, \sigma)\).
As a cautionary note, the differential form of the governing equations, e.g. eqs. (3.40) - (3.41), are predicated on smoothness of the system. Although it is possible and arguably more elegant to reformulate the governing equations starting from eq. (3.50) as integral equations over segments that do not \textit{a priori} require smoothness \cite{17}, one encounters the apocryphal “conservation of difficulty” – i.e. the difficult cannot be made simple by mere manipulation, one only changes the difficulty’s location. Accurately determining the $I_1$ and $I_2$ values for a non-smooth system requires complete river geometry, and even then the relationship between the geometry and frictional losses becomes problematic. Furthermore, even when the system of equations can be accurately developed, the resulting set of nonlinear equations are usually only efficiently solved with methods that require $C^1$ smoothness of the system (see §5.6).

### 3.8 Friction slope

The momentum equations in §3.7 use a friction slope ($S_f$) as a model term. The principal goal of $S_f$ is to represent the turbulent losses associated with the Reynolds stress terms integrated over the channel cross-section (see also §4.5). Comparing eq. (3.34) to (3.37) and using (3.17) we find that

\begin{equation}
S_f = \frac{1}{ga} \int_a \frac{\partial}{\partial x_j} u_j u'_j da
\end{equation}

Using theory, one can draw relationships between the Reynolds stresses and boundary roughness in a flow to estimate the friction slope, e.g. \cite{13, 44}. Thus, in theory it should be possible to design an \textit{a priori} correct model for $S_f$. However, in practice $S_f$ is generally represented with an empirical term that is adjusted (calibrated) to improve the match between measured data in a real river and the model, e.g. \cite{23}. Calibration presents a conceptual problem for the $S_f$ term, as the Saint-Venant derivation introduces a variety of errors, including approximations in the nonlinear term ($\beta$, §3.5), in the pressure force term (hydrostatic, §3.6), in neglect of channel curvature (§3.4), and geometric approximations for changing channel morphology in streamwise integration of the...
3.9. Summary

Thus, if the model solution uses a calibrated $S_f$ to match the physically measured values in a real river, the errors associated with model approximations must be inherently wrapped into the $S_f$ term. That is, $S_f$ must represent not only the Reynolds stresses, it must also represent the non-hydrostatic pressure gradients, the non-uniformity of the velocity field, and the along-channel topographic effects. We do not have a comprehensive theory for how all these features can be combined into a single empirical coefficient, so parameterizing $S_f$ is an important source of uncertainty for any river model (see [37] for further discussion of model uncertainty).

The most common model for the friction slope is derived from the Chezy-Manning equation, introduced in §1.5.2 and repeated here as

$$Q = \frac{1}{n} a R_h^{2/3} S_f^{1/2}$$

where $R_h$ is the hydraulic radius ($a/P_w$) defined from the wetted perimeter $P_w$ and the channel cross-sectional area. As shown by [52], it is convenient to define an “equivalent friction geometry” $F$ in place of $R_h$, as

$$F(a) = \frac{P_w^{4/3}}{a^{7/4}}$$

such that

$$ga S_f = gn^2 F Q^2$$

Implementation in the governing equations, e.g. (3.45) provides

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \beta \frac{Q^2}{a} \right) + ga \frac{\partial h}{\partial x} = g \left( a S_0 - n^2 F Q^2 \right)$$

which shows that frictional effects complicate the governing equations through introduction of a nonlinear source term that depends on the channel roughness parameterization ($n$), channel geometry ($F$), and the flow rate ($Q$).

3.9 Summary

The above presents the complicated process to get from the underlying 3D equations based on first principles (Navier-Stokes), to the tractable
1D Saint-Venant equations that are suitable for river networks. The 1D equations should be considered the fundamental equations for modeling river network flows where variations in surface elevation (§4.6) are both unsteady and important. Simpler river models (e.g. the energy model §4.5 or Chezy-Manning, §1.5.2) have better direct analogies to electric circuits because flood waves and gravity waves are neglected, but these are of limited use for large-scale river network models [37].

An important point in the above derivation is that all the errors introduced in the process of simplification and approximation are typically wrapped into a friction term \( S_f \) that represents energy losses along the channel (§3.8). Modeling and calibrating the frictional losses is an empirical challenge for river network modeling.

In the context of real-world problems, it should be emphasized that the governing equations for fluid flow are partial differential equations (PDE) developed from continuum mechanics, hence assuming smoothness and continuous differentiability that do not actually exist throughout a river network (e.g. §4.11). Thus, a simple PDE model alone cannot simulate a real river network; a 1D Saint-Venant simulation scheme must include sub-models of non-continuous physics discussed in Chapter 4. In addition, a modeler must also carefully consider when not to apply the 1D equations; e.g. the dynamic behavior of a river overbanking into a floodplain is poorly modeled in 1D, and is best represented by either 1D-2D hybrids or full 2D models (§4.12).
4

A River Network Primer

4.1 Introduction

To the extent that a river can be conceptualized as a simple DAG network (§1.2), it might appear the underlying problem is solved by merely identifying the appropriate equations along branches and between nodes (i.e. as discussed in Chapter 3). However, modeling a natural system has its peculiarities and uncertainties associated with non-linear interactions of the physical processes and our inability to completely quantify the boundary conditions, as introduced in §1.4. The following provides a review some of the key physical features that complicate river analysis, and provides an introduction to some of the different ways that hydrologists and hydraulic engineers have quantified river behavior. Although we attempt to draw some analogies between river physics and electric circuits – the open questions here are numerous. It is our hope that the following discussion will inspire new insights by others.
4.2 Boundary conditions

For simple electric circuits, the given voltage provides the external boundary condition that impels a current. If we consider only a river running from a lake down to the ocean, the river/electricity analogy is quite strong: the difference between the lake elevation and the ocean elevation is the potential energy difference that drives the flow through the river channels. However, for a river network in a broader sense, the potential energy is unwieldy when applied to the wide variety of boundary conditions: landscape runoff from rainfall, flow from underground springs, seepage into groundwater aquifers, and melting of snow/glaciers. Rivers are merely one piece of the hydrological cycle, as water is evaporated from lakes and oceans, transpired from plants, transported through the atmosphere, precipitated onto the landscape, stored in frozen form, slowly transported through surficial aquifers, and eventually reaches the rapid transport channels of streams and rivers.

In a river network model, the principal boundary conditions are flow rates from the landscape into the channel, which occur both at the ultimate headwater (where a stream initially coalesces from groundwater inflows) and along a river’s path to the ocean as the landscape contribute flows as lateral boundary conditions. Thus, the equivalent for an electric circuit would be a boundary condition on current rather than voltage. The river boundary flows are necessarily the outputs from hydrological models representing runoff, groundwater fluxes, and atmospheric exchange. Unfortunately, the coupling is not one-way, as the river surface water elevations are boundary conditions required by a hydrological model for accurate estimation of the flow rates to the river. Fortunately, the landscape runoff responds to the time-integrated effect of the river flow over longer time scales so hydraulic and hydrologic simulations can run at different time steps, which provides greater computational efficiency. Hydrological simulations can often run on time steps of hours to days due to the slow transport rates and lagging response to impulse events. In contrast, river models typically need time steps of minutes. A consequence of the coupled physical processes in the real world is that a river network model generally cannot be tested against
observed data without a companion hydrological model for boundary conditions.

4.3 Normal depth and initial conditions

One might readily intuit that if water is poured into an infinite-length open channel of uniform geometry and slope at some constant rate $Q$, then the water depth should also be uniform throughout the channel (away from the pour point). This is the necessary depth to obtain a balance between the down-slope gravitational force and up-slope frictional force so that the flow rate is constant all along the channel (i.e. $\Sigma$ forces = 0). This ideal depth is known as the “normal” depth, and is derived from the Chezy-Manning equation (§1.5.2). Computing the normal depth is analogous to computing the diameter of a wire of a particular material that carries a given current for a given voltage.

The normal depth is often thought of as the expected water depth ($h$) for steady flow in a channel that does not have any spatial gradients in $a, n, R, Q, \text{ or } S_0$. For known geometry relationships between depth and cross-section shape, $h : a$ and $h : R$, there will be a unique normal depth that satisfies the Chezy-Manning equation, eq. (1.10). For example, consider a simple rectangular channel where the cross-sectional area $a = Bh$, where $B$ is the channel breadth and the wetted perimeter $P_w = B + 2h$, so that the hydraulic radius is $Bh (B + 2h)^{-1}$. The Chezy-Manning equation reduces to

$$Q = \frac{1}{n} Bh \left( \frac{Bh}{B + 2h} \right)^{2/3} S_0^{1/2} \quad (4.1)$$

For given $Q, n, B$ and $S$, the solution for $h$ provides the normal depth. In practice, the normal depth at flow rate $Q$ is the condition the flow will tend towards over time in the absence of spatial or temporal perturbations to the flow.

Unsteady flow simulations for a river network require initial conditions for the water surface elevation and flow rate throughout the system. Such synoptic data are impractical to obtain, so the normal depth provides a useful approximation: given initial $Q$, Manning’s $n$, channel geometry $h : a$, $h : R$, and $S_0$ throughout a river network, we
can use the normal depth that satisfies Manning’s equation, eq. (1.10,) to initialize a model.

Because the normal depth solution is only an approximate initial condition, a river network model requires some “spin-up” time. That is, some initial simulation time is required to effectively dilute the errors associated with the approximate initial condition. Although the spin-up time for some fluid mechanics systems can be a priori estimated based on physical processes, for a river network the spin-up time scale remains an open question. As a practical matter, the spin-up for a particular system can be determined empirically by a series of unsteady simulations starting at different times. At the simulation time when all the simulations are producing the same results, then the initial condition effects have been diluted and the simulation reaching this point in the shortest time provides the spin-up time.

The concept of “spin-up” is similar to the “source-ramping” technique used in electric circuit simulation. In source ramping all boundary conditions (e.g., external voltage sources) are set to zero. The circuit solution can be easily calculated as all branch currents and all nodal voltages are zero. The voltage sources (boundary conditions) are then gradually “ramped” up at a selected rate until they reach desired operational levels. The final solution from source ramping is used as an initial condition of the circuit. For circuits with multiple initial conditions, sophisticated methods such as homotopy can be used [80].

4.4 Observed water elevation and flow relationships

“Stage-discharge” graphs are a common approach to quantifying observed river behavior [59]. In hydraulics jargon, the water elevation at some point above a known local datum is called the stage, while the flow rate is known as the discharge. This relationship between the gravity potential and flow rate is equivalent to graphing the evolving voltage-current relationship at some point in a circuit.

Despite a range of nonlinearities in the governing equations (Chapter 3), the stage-discharge behavior at a particular point in a river tends to be log-linear. Historic measurements of flow and elevation can
be used to create stage-discharge graphs, which are commonly available at selected locations along major rivers. Such graphs only define the conditions at a single point (Fig. 4.1) and cannot be used with any reliability away from that point [22].

During floods, established stage-discharge graphs can be used to estimate the flow rate in a river [16]. By accumulating observed flow rates in upstream river branches and predicting downstream travel time in each branch, rough estimates can be developed for the expected future flow rate and expected “flood stage” at a downstream point. Such estimates are generally accurate when floods fall within the set of previous observations that were used to develop the stage-discharge graph. However, extreme flood conditions may be outliers of an established relationship. Furthermore, erosion/aggradation or other changes to a river can cause changes to the stage-discharge relationship that invalidate an established graph. For example, a new highway bridge built downstream of a historic measuring point could significantly change the stage-discharge relationship: the bridge increases the frictional resistance, leading to a greater backwater effect (§4.6) such that a lower discharge is correlated with a higher stage.

Stage-discharge relationships provide data that can be used to calibrate or check a river network model’s performance. However, such data must be used with caution as they are not always updated with every significant change in the river.

### 4.5 Energy description of flow

The differential river dynamics equations developed by Saint-Venant (§1.5.3) imply smoothly varying properties along the river. Unfortunately, there are many natural features and man-made structures in a river where such approximations cannot hold. A common approach for modeling these features is through an energy conservation equation. Over short distances and time scales, we can write a steady-state mechanical energy equation between an upstream point \( e_1 \) and a downstream point \( e_2 \) as

\[
e_1 = e_2 + e_L
\]
Figure 4.1: Stage discharge curve for River Ibicuí at Alegrete (upper) and Manoel Viana (lower) in the Amazon basin illustrating the difference in observed relationships at different points in the river. [56]
4.5. Energy description of flow

where energy losses \((e_L)\) represent the irreversible conversion of mechanical to thermal energy via vicious forces. Mechanical energy and work can be further separated into three components, kinetic, potential, and pressure. The kinetic energy \(\left(\frac{1}{2}mu^2\right)\), where \(m\) is mass and \(u\) is velocity, is the energy contained in the fluid’s velocity. The potential energy \(mgz\) is the energy of the fluid by virtue of its elevation, \(z\). Pressure \((p)\) can considered a measure of the work that can be potentially be done by a column of water. Energy per unit weight has units of length, which is known as “hydraulic head,” or simply head. The relative energy loss per unit length in a river is represented by the head loss per unit length, also known as the “friction slope” \((S_f)\). For two cross sections on the river separated by a distance \(L\) the head loss will be

\[
h_L = \frac{e_L}{\rho g} = \int_0^L S_f dx \tag{4.3}
\]

So if \(S_f\) is taken to be spatially uniform over a short river reach we have

\[
\left(\frac{p}{\rho g} + z_b + \frac{u^2}{2g}\right)_1 = \left(\frac{p}{\rho g} + z_b + \frac{u^2}{2g}\right)_2 + S_f L \tag{4.4}
\]

where \(z_b\) is the channel bottom elevation. With the hydrostatic approximation (§3.6) the pressure is due only to the depth \((h)\) of water, so that

\[
h_1 + z_{b(1)} + \frac{u_1^2}{2g} = h_2 + z_{b(2)} + \frac{u_2^2}{2g} + S_f L \tag{4.5}
\]

This equation can be understood graphically, as shown in Fig. 4.2, where the three lines represent the channel elevation, the water surface elevation and the total energy in the system. The latter two are known as the Hydraulic Grade Line (HGL) and the Energy Grade Line (EGL). The distance between the water surface and the EGL is a graphical representation of the kinetic energy of the flow, such that the slope of the EGL is simply \(S_f\). The distance between the channel bottom and the water surface (depth) is a graphical representation of the hydrostatic pressure available in the system. The bottom elevation is the base level of potential energy at a point.
This approach allows us imagine river flow as trade-offs between three forms of hydraulic head: depth ($h$), bottom elevation ($z_b$) and velocity head ($u^2/2g$), with head losses ($S_fL$) decreasing the total head (energy) available as the river flows downstream. Thus, as a river channel becomes narrower downstream, the flow ($Q$) may be carried by increasing $h$, which requires decreasing $V$ to compensate for the increased $h$. Counter-intuitively, it is also possible that a narrower channel may result in a decreased $h$ and increased $u$. That is, the same $Q$ can be carried in a single river channel at a multitude of different values of $h$ and $u$. What condition actually occurs will depend on the relationship between the bottom slope and the head losses over the section, with the latter a function of both flow rate and geometry. This behavior illustrates one of the complicating features of river flows that does not have an analogy in electric circuits – the water depth, and thus cross-sectional area of the “conductor” is a nonlinear function of the geometry and flow.

The energy description of river flow is useful for evaluating flow conditions where the Saint-Venant equations do not hold. The energy equations cannot entirely replace the Saint-Venant equations because they do not account for momentum effects with temporal evolution. Implicit in the energy equations is a presumption that the flow being
4.6 Flood waves and backwater waves

Any local impulse in a smoothly-flowing river will cause disturbances to propagate both upstream and downstream. Imagine that a large volume of water is rapidly introduced at some point in an otherwise smooth river. Such an impulse occurs when a flooding tributary meets the main stem of a river (or vice versa). The impulse causes a rapid rise in the water surface elevation, which causes the downstream surface gradient to become a larger negative number ($\partial \eta / \partial x < 0$ where $x$ increases downstream) while the surface gradient upstream becomes a smaller negative number. The larger-magnitude downstream gradient will increase the flow rate downstream of the impulse, while the reduced-magnitude gradient upstream will decrease the upstream flow rate into the impulse area. Note that the introduction of the impulse volume does not necessarily reverse the flow in the river\footnote{Flow reversals are actually possible during floods; indeed, they commonly occur whenever the flood wave of a large river such as the Mississippi, meets a smaller tributary.}, but it generally slows the upstream flow rate.

The forward and backward propagation of the impulse’s effect can be understood by considering Saint-Venant’s eqs. (1.11) and (1.12) written for a rectangular channel of breadth $B$ and depth $h$ where $a = Bh$ and $h = \eta - z_b$. Here $z_b$ is the channel bottom elevation, taken to be uniformly zero (no bottom slope) for simplicity. For an inviscid (frictionless) flow we obtain

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0 \quad (4.6)$$

$$\frac{\partial h}{\partial t} + h \frac{\partial u}{\partial x} + u \frac{\partial h}{\partial x} = 0 \quad (4.7)$$

We consider a change of variables as $c = \sqrt{gh}$, and let $\alpha_p \equiv u + 2c$ and $\alpha_m \equiv u - 2c$. Adding eq. (4.6) to eq. (4.7) or subtracting the latter terms can be neglected. Such an assumption does not hold when surface waves have significant effects.

4.6 Flood waves and backwater waves

described is steady-state, or sufficiently close to steady state that $\partial / \partial t$
from the former provides hyperbolic equations:

\[
\frac{\partial \alpha_p}{\partial t} + (u + c) \frac{\partial \alpha_p}{\partial x} = 0 \tag{4.8}
\]

\[
\frac{\partial \alpha_m}{\partial t} + (u - c) \frac{\partial \alpha_m}{\partial x} = 0 \tag{4.9}
\]

The velocity \( u \) is the downstream water velocity, whereas \( c = \sqrt{gh} \) can be shown to be the approximate velocity ("celerity") of a shallow-water gravity wave; i.e. a wave motion traveling on the water surface whose return to equilibrium is forced by gravity (similar to a pendulum). Thus, eqs. (4.8) and (4.9) can be interpreted as the Lagrangian transport of characteristics \( \alpha_p \) and \( \alpha_m \) at the net velocities of \( u + c \) and \( u - c \). For \( u < c \), eq. (4.9) implies a disturbance propagating upstream, hence a "backwater" wave. For all conditions eq. (4.8) implies a downstream "forerunner" wave propagating at speed \( u + c \). These waves propagate both with and against the flow, and affect the upstream and downstream flow behavior. These are another complicating feature for river flows that does not have an analogy in electric circuits. Indeed, we have been unable to develop even a similarity idea to illustrate an equivalent effect.

Both backwater and forerunner waves are "dynamic" waves that arise from the force balance in the momentum equation. However, the bulk of the flow moves at the "kinematic" wave speed, which is essentially an effect of continuity, i.e. the volume balance of the fluid moving downstream [79]. The term kinematic is used to indicate solution of motion without consideration of forces. The kinematic wave speed can be approximated as \( dQ/da \), which is less than \( u + c \) as long as \( u < c \) (known as a subcritical flow, §4.7). A common approach to modeling river networks is to use a "kinematic wave approximation," which effectively eliminates the dynamic (momentum) waves from the solution. River network models with this approximation are common and relatively easy to create. However, they cannot model the important dynamic backwater effects that play a critical role in flooding rivers and in the upstream effects caused by downstream changes in channel characteristics. As such, kinematic wave approximations will not be considered further in this paper.
4.7 Subcritical and supercritical flow

Because of dynamic waves (§4.6), information in a river network can generally propagate both upstream and downstream. Downstream, information moves at the sum of the water velocity and the wave velocity \( u + c \). Upstream, information propagates only at the velocity difference \( u - c \), so there is a change of equation behavior when \( u \geq c \) as the upstream propagation of information will cease. Under such conditions the impulse flood wave described in (§4.6) will have no upstream effects and will simply propagate downstream. In common hydraulic parlance, the condition where \( u = c \) is known as “critical” flow, which corresponds to a wave trying to move upstream on the surface of the water at exactly the same speed as the water is advecting the wave downstream. At critical flow, any wave disturbance simply stands in place. When the water velocity further increases, such that \( u > c \), the flow is deemed “supercritical” so that any wave disturbance is simply swept downstream. The common condition for most rivers is “subcritical,” where \( u < c \). The relative criticality of the flow is defined by the non-dimensional Froude number

\[
Fr = \frac{u}{\sqrt{gh}}
\]

so that \( Fr > 1 \) is supercritical, \( Fr < 1 \) is subcritical, and \( Fr = 1 \) is critical. Using \( Q = ua \), critical depth is then

\[
h_c = \frac{Q^2}{ga^2}
\]

For rectangular channels where \( a = Bh \), computing \( h_c \) is trivial; but the matter is nontrivial in a natural river where \( a = f(h) \) is only known from empirical measurements of \( a : h \).

The ideas of supercritical and subcritical flow can easily lead to confusion, as there is a natural tendency to equate “super” with “big” and thus erroneously think that supercritical implies a big or deep river. Instead, supercritical flows are in shallow rivers; i.e. when \( h < h_c \) the flow is shallow, fast, and supercritical. In contrast, when \( h > h_c \) the flow is deep, slow, and subcritical. Long reaches of supercritical flow typically only occur in steep mountain rivers and streams. However, smaller
rivers and streams will sometimes exhibit shorter “riffle-pool” sequencing where near-critical or localized supercritical conditions may be expected in short, shallow riffle sections, followed by deeper slow-moving pools [64]. The possibility of supercritical flows over even small segments has an impact on numerical model stability for river networks. A river reach with supercritical conditions requires two upstream boundary conditions and will not support a downstream boundary condition [29, 55]. In contrast, a subcritical river reach requires a single upstream and a single downstream boundary condition.

4.8 Cross-section Surveys

The representation of poorly-known channel cross-section geometry and its interaction with empirical frictional resistance models ($S_f$) is a principal driver of uncertainty and error in river network models. Bathymetric surveys in full 3D using either green lidar\(^2\) or multi-beam sonar are becoming more common over short river reaches; however it is unlikely that we will ever have a comprehensive 3D bathymetric data set for the millions of river miles in a continental river network, so issues of cross-section survey accuracy and spacing will continue to be important [37].

River geometry is conventionally defined by cross-section surveys, which provide a relationship of bottom elevation, $z_b$, against a cross-channel distance, $y$ [10]. The $y$ distance\(^3\) is typically either measured towards either bank from the deepest point in the channel (known as the “thalweg”) or from one of the river banks. From the standpoint of river network modeling (§5.6), the details of the survey are relatively unimportant as long as the relationships between $h : a$ and $h : P_w$ can be determined and $z_b$ are either measured from or can be converted to a consistent absolute datum. Unfortunately, river surveys are not consistently referenced to a single datum. Ideally, all survey data in the

---

\(^2\)Lidar was originally written as LiDAR, to indicate an acronym for Light Detecting and Ranging. As with its cousins sonar and radar, lidar has now entered the lexicon as a normal English word without special capitalization.

\(^3\)Nota bene: in many hydraulics textbooks $y$ is used as a measure of depth instead of as a coordinate axis.
USA, Canada and Mexico should be referenced to the North American Vertical Datum of 1988 (NAVD88).

River models generally do not directly use \( y : z_b \) survey data, although these are required input data. As a model pre-processing step, the \( y : z_b \) relationships are converted into approximate relationships of area with depth, \( h : a \), and hydraulic radius, \( R_h : a \) (see §1.5.1 for discussion of \( R_h \)). Alternatively, [52] showed an \( F : a \) relationship in place of \( R_h : a \) (see §3.8).

A principal unresolved problem is the uncertainty associated with whether or not two adjacent cross-sections provide a reasonable representation of the channel geometry between them. Survey data typically provide a simple slice of some position \( x_i \) along the river, with no information of how typical that slice might be. Survey positions \( x_{i+1} \) and \( x_{i-1} \) on either side of \( x_i \) may have significantly different cross-sectional geometry [78]. Although ideal surveys may be conducted with expert selection of locations to capture representative cross-sections at regular intervals, more commonly survey locations are limited by accessibility, safety of survey team, and budgetary constraints. The river network modeler has relatively little control or knowledge regarding the adequacy of the survey in representing the geometry between cross-sections.

As discussed in [52, 50], the smoothness of the \( y : z_b \) relationship as translated into the model can affect model performance. Generally, non-smoothness is associated with sharp features in the cross-sectional shape. Such geometric features tend to be localized and give rise to non-uniformities in the flow field whose scales are less than the channel width. Because 1D models are developed by integrating over the channel cross-section (§3.5), these smaller flow features are below the model resolution. This is not to say that such features are unimportant; however, their contribution should be (ideally) represented through the non-uniformity parameter \( \beta \) (see §3.5.1) and through frictional resistance (e.g. Manning’s \( n \), §3.8). We should like to be able to relate the shape of channel cross-section directly to \( \beta \), but to date there is no consistent theory and most modelers simply take \( \beta = 1 \) as an adequate approximation. Thus, the effects of non-uniformity and smaller
A River Network Primer

features are wrapped into $n$ and subject to model calibration \[77\]. This uncertainty gives the modeler some leeway in interpreting survey data. Although the general behavior of $h : a$ and $F : a$ relationships should be maintained, sharp features that cause numerical problems usually can be smoothed \(\S 5.6\). However, we presently lack any quantitative understanding of how much smoothing can be accomplished without affecting model solutions. There are similarities between the problems of uncertainties and smoothing of river cross-sections and the uncertainty and model simplification of VSLI conductors \(\S 2.6\).

4.9 Unsurveyed rivers

Unsurveyed rivers present a common problem for river network modeling. In many cases, we can approximate river geometry using historical stage-discharge data \(\S 4.4\) and the approximate river “bank full” width\[4\] The latter may be digitized from satellite or lidar data, whereas the former are typically developed by government agencies over decades of surveys and analysis.

The concepts of “hydraulic geometry,” can be used model the relationships between the channel breadth, flow rate, mean depth , and mean velocity as power law functions. As proposed by \[49\], these functions are:

\[
\begin{align*}
w &= aQ^b , \\
    d &= cQ^{f} , \\
    v &= kQ^{m} \\
\end{align*}
\]

(4.12)

where the nomenclature of $w$ for channel width, $d$ for mean channel depth, and $v$ for mean water velocity are standard in the hydraulic geometry literature (associated with hydrology and geosciences), while the hydraulic literature (associated with civil engineering) typically uses $h$ for depth and reserves $w$ for a vertical component of the water velocity while using $b$ or $B$ for channel breadth. In eq. (4.12) the empirical coefficients $a, b, c, f, k, m$ are such that $ack = 1$ and $b + f + m = 1$. Note that these functions are developed by observation of real channels and are the result of how the range of flows acting over decadal or longer

\[4\]The bank full width is the river width just before it overtops its banks into the flood plain.
time-scales cause erosion and aggradation of the river geometry. That is, the flow rates are a forcing condition for the long-time evolution of the river channel, whereas over the shorter time-scales of a river network model the channel geometry is a forcing condition on the flow behavior.

An example of the utility of the hydraulic geometry approach can be found in [6], where Landsat satellite imagery was used with historical flow data from 82 gaging stations with 15 years of data to estimate the hydraulic geometry coefficients for the entire Amazon River basin, which covers $6 \times 10^6$ km$^2$ of landscape and $4.4 \times 10^6$ km of rivers and streams (based on 1 km$^2$ resolution). Because of the uncertainty inherent in obtaining empirical coefficients for hydraulic geometry, a common approach for representing the river cross-section is to simply assume a rectangular channel with the width and depth of bankfull conditions [58]. With a rectangular channel, once the breadth $B$ is defined the $h : a$ relationship required for solving Saint-Venant’s equations (§1.5.3 §3.7) is simply

$$h = \frac{a}{B}$$  \hspace{1cm} (4.13)

while the $R_h : a$ relationship is

$$R_h = \frac{aB}{2a + B^2}$$  \hspace{1cm} (4.14)

Thus, with sufficient data and appropriate simplifications it is possible to provide estimated cross-section geometry for unsurveyed rivers. However, it should be recognized that errors in the cross-section description will be calibrated into the friction representation (e.g. Manning’s $n$).

4.10 Tributary junctions

A junction of two or more rivers or streams is inherently a source of uncertainty and error in 1D models. The underlying presumption for 1D is that we can integrate the momentum equations across the channel (§3.5) because the streamwise velocities are significantly larger than the cross-stream velocities. However, where two rivers join there will
always be significant cross-stream velocities \cite{20}. Furthermore, describing the cross-section geometry at a river junction is problematic, as it is difficult to precisely define a single point where two rivers become one. This problem is not unique to river models, as the failure of simple KVL relationships at conductor bends and joints is a directly analogous (§2.6). To date, we have not attempted to exploit methods in VLSI for the treatment of channel junctions.

To enforce continuity, eq. (1.11), we imagine a junction to be a single point in space, such that it cannot store volume. It follows that an internal boundary condition at a junction is

\[
\sum Q_{in} = \sum Q_{out}
\]  

(4.15)

where \(Q_{in}\) are the inflows to the junction and \(Q_{out}\) are the outflows from a junction. A typical junction is where two upstream rivers become a single downstream river; three or more upstream rivers joining at a single point are possible, but relatively rare. However, when one upstream river is flooding, it is possible that backwater effects (§4.6) driven by the flooding branch can eventually result in a reverse flow in the non-flooding branch. Thus, the case of a single upstream branch and multiple downstream branches is possible and cannot be discounted. Furthermore, if a network model is extended into a river delta, the branching from a single upstream into multiple downstream branches will be a common feature.

A second internal boundary condition at a junction is continuity of the free surface elevation, i.e. there is only one value of \(\eta\) possible at a free-flowing junction (however, see §4.11). Because our solution methods are for \(a\) and \(Q\), with \(\eta\) (or \(h\)) as an auxiliary variable, direct imposition of an \(\eta\) (or \(h\)) boundary condition is inherently nonlinear, which can cause convergence problems. In \cite{52}, a method is outlined to use \(\eta : a\) relationships for upstream and downstream branches at a junction to form a set of pre-processed auxiliary equations so that cross-sectional area of the \(i^{th}\) upstream branch, \(a_{u(i)}\), is related to the area of the \(k^{th}\) downstream reach, \(a_{d(k)}\), for a surface elevation of \(\eta\) by a local linear relationship:

\[
a_{u(i)}(\eta) = r_{(i)}a_{d(k)}(\eta)
\]  

(4.16)
where \( r_{(i)} \) provide the geometric relationships for the cross-sections. With this approach, enforcing continuity of the free surface is a simple linear condition if the time \( n + 1 \) and \( n \) values of \( r_{(i)} \) are sufficiently close, which should be true for small changes of \( \eta \).

**4.11 Dams, lakes, and hydraulic structures**

Both nature and man routinely interfere with the free flow of water in a river network. Such interference often means the 1D open-channel flow conditions of the Saint-Venant equations (§3.7) are poor approximations of local physics. Thus, such features require internal boundary conditions that enforce appropriate fluxes out of an upstream river segment and into a downstream river segment. Lakes and waterfalls are the dominant natural features in this category. Reservoirs created by flood-control dams are the most obvious man-made features, but a wide range of smaller features are also scattered through the landscape. Weirs, gates, culverts and locks are some of the smaller hydraulic structures used for various types of flow control (Fig. 4.3). All of these features can have discontinuous water surface elevations; i.e. mathematically the upstream and downstream \( \eta \) have smooth, one-sided derivatives joined by a Heaviside step function across the structure.

From a physics perspective, these river network elements have a common behavior: they provide a hydraulic control that changes the relationship between \( Q : \eta \), such that Saint-Venant eqs. (1.11) and (1.12) cannot be applied. Large flood-control and water-supply dams will have standard operating procedures setting the outflow as a function of a myriad of possible factors, including the seasonal cycle, inflow rates, reservoir water level, downstream demand, expected rainfall, or an expectation of drought conditions. Accurate river network modeling requires creating a customized operational model of \( Q : \eta \) for each structure. Obtaining and interpreting such data is a non-trivial aspect of operational modeling of large-scale river networks.

Weirs are the most common form of small dams, which are installed for local as flood control, agricultural water supply, flow regulation, or limit upstream estuarine salt-water intrusion. A weir is essentially a
Weir that allows water to flow over the top under normal flow conditions [15]. Some weirs have control gates to allow controlled flow of impounded water when the water level drops below the upper weir edge, but these are less common. In typical operation, weir flow is governed by equations of the form

\[ Q = \alpha (\eta_h - z_w)^\beta \]  \hspace{1cm} (4.17)

where \( \eta_h \) is the headwater (upstream) water surface elevation, \( z_w \) is the elevation of the weir upper edge so that \( \eta - z_w \) is the depth of the water over the weir, \( \alpha \) is a dimensional coefficients depending on weir geometry (usually including a factor of \( \sqrt{2g} \)), and \( \beta = 3/2 \) is the typical exponent. However, it should be noted that the standard coefficients in the literature may also be functions of \( \eta_h - z_w \) that depend on the specific weir geometry, which may further complicate the formula. For
4.11. Dams, lakes, and hydraulic structures

Example, the flow over a rectangular sharp-edged weir can be modeled as [71]

\[ Q = \frac{2}{3} \sqrt{2g} C_d B (\eta_h - z_w)^{3/2} \]  \hspace{1cm} (4.18)

where \( B \) is the cross-channel breadth of the weir and \( C_d \) is the non-dimensional drag-coefficient given by

\[ C_d = 0.611 + 0.08 \left( \frac{\eta_h - z_w}{z_w - z_b} \right) \]  \hspace{1cm} (4.19)

with \( z_b \) representing the elevation of the channel bottom upstream of the weir. Exact numerical implementation of this equation would be

\[ Q = \alpha_1 (\eta_h - z_w)^{\beta_1} + \alpha_2 (\eta_h - z_w)^{\beta_2} \]

\[ \alpha_1 = (0.407) \sqrt{2g} B \]
\[ \alpha_2 = \left( 5.33 \times 10^{-2} \right) \sqrt{2g} \left( \frac{B}{z_w - z_b} \right) \]  \hspace{1cm} (4.20)
\[ \beta_1 = 3/2 \]
\[ \beta_2 = 5/2 \]

However, because \( C_d \) is an empirical factor that is fit to experiments, it cannot be considered precise. Using a Taylor-series argument, as long as \( d\eta/dt \) is small, we can use a time lagging approach such that the flow at time step \( n + 1 \) is reasonably represented as

\[ Q^{(n+1)} = \frac{2}{3} \sqrt{2g} C_d^{(n)} B (\eta_h^{(n)} - z_w)^{3/2} \]  \hspace{1cm} (4.21)

If the time step is sufficiently small, a lagged linearized approximation can be written as

\[ Q^{(n+1)} = \frac{2}{3} \sqrt{2g} C_d^{(n)} B \sqrt{\eta_h^{(n)} - z_w} \left( \eta_h^{(n+1)} - z_w \right) \]  \hspace{1cm} (4.22)

In comparing to eq. (4.20), we find eq. (4.22) is more computationally-friendly as it represents a linear internal boundary condition between upstream and downstream reaches governed by the Saint-Venant equations.

A weir is functioning normally when the headwater is above the weir edge and the downstream water (tailwater, \( \eta_t \)) is below the weir.
edge. When the tailwater rises such that $\eta_t > z_w$, the weir is said to be submerged. Submerged weirs will not follow eq. (4.17), but instead should be considered part of a Saint-Venant solution as $\eta$ is once again continuous. A frictional model can be used to account for the effects of the submerged weir (§3.8). Because of the equation change between a submerged and unsubmerged weir, the friction model (e.g. Manning’s $n$) and the weir coefficients must be formulated to smoothly transition between regimes such that the one-sided derivatives from upstream or downstream are continuous at the weir through the transition. Step function transitions whose derivatives are discontinuous may result in unphysical oscillations and slow convergence as the solution switches between equations. Hydraulic engineering references (e.g. [45]) have more complex forms of submerged weir equations that can be used to transition between the Saint-Venant equations and normal weir flow. However, it is not clear whether blending an additional equation between the Saint-Venant and equations of the form of eq. (4.17) will significantly improve model accuracy.

Submerged gates may be part of dam, weir or lock structures, with complex operating conditions and flow equations. For many structures, the flow rate can be related to headwater and tailwater surface elevations in the general form similar to eq. (4.17)

$$Q = \alpha (\eta_h - \eta_t)^\beta$$ (4.23)

where $\alpha, \beta$ are again dimensional coefficients depending on geometry and flow rates.

Culverts are used to direct flow beneath roads or other structures when flow rates and channel geometry make bridging an unnecessary expense. Culverts are challenging because their behavior changes as a function of whether they are “inlet” or “outlet” controlled. Briefly, an inlet controlled culvert can be thought of as having the water depth upstream of the culvert controlling the flow rate. For outlet control, it is the water depth on the downstream side controlling the flow rate. Inlet control takes on the general form

$$Q = (\alpha h_h - \beta h_c)^{1/2}$$ (4.24)

where $h_h$ is the headwater depth, i.e. $\eta_h - z_b$; $h_c$ is the critical depth
from eq. (4.11) for the flow rate $Q$, and $\alpha, \beta$ are dimensional coefficients that depend on culvert geometry and flow conditions. Note that when the culvert inlet is fully submerged the equations simplifies with $\beta = 0$. For outlet control, both the headwater and tailwater surface elevations play a role as

$$Q = \alpha (\eta_h - \eta_t)^{1/2} + \beta$$

(4.25)

More detail on culvert equations and calculating values for $\alpha, \beta$ for different culvert geometries and flow conditions can be found in [71] as well as in a variety of hydraulic handbooks. Note that the $\alpha, \beta$ in eqs. (4.17) - (4.25) are not related or even of the same dimensional form. These are simply placeholders for the detailed geometrical and flow relationships developed through empirical studies.

Bridges provide particular problems for river flow as the channel width is typically constrained and bridge piers/obstructions decrease the flow cross-sectional area and increase the flow resistance. Significant energy losses may occur at bridges, typically leading to backwater effects (§4.6). As with other structures, the sharp spatial flow gradients near a bridge will violate the approximations used for deriving the Saint-Venant equations, so ideally a bridge should be treated as an internal boundary condition in a river network solution. Under normal flow conditions, flow through bridges takes on the general form dependent on headwater and tailwater elevations as

$$Q = (\alpha \eta_h - \beta \eta_t)^{1/2} + \gamma$$

(4.26)

where $\alpha, \beta, \gamma$ are dimensional coefficients that are functions bridge geometry and flow conditions. Because the depth of submergence affects the flow area beneath the bridge, the $\alpha, \beta, \gamma$ coefficients are generally functions of $\eta_h$ and $\eta_t$.

During extreme floods, upstream water levels may reach the bridge deck, which will cause a bridge to act like a large culvert with inlet control. When both the upstream and downstream edges of the bridge deck are submerged, it behaves similar to flow through a culvert with overtopping. Thus, like a culvert model, a bridge model must be able to smoothly transition between model equations during the river network solution. There are a wide variety of bridge models that can be used
and a wide variety of bridge geometries that must be handled. The methods outlined in [71, 10] provide good starting points for detailed model development.

Lakes, both natural and man-made, slow the velocities in a river. The flow in a lake is complex and multi-dimensional [39], so modeling the momentum dynamics with the 1D Saint-Venant equations is of questionable utility. For river network modeling, most natural lakes can be modeled as a simple point relationship of the lake volume \( V \) to its water surface elevation, i.e. \( V : \eta_L \). Lake morphometry is generally available in “hypsographic curves,” which provide the surface area of a lake \( A_L \) as a function of \( \eta_L \). Thus, a simple volumetric lake model is

\[
\frac{d\eta_L}{dt} = \frac{Q_{in} - Q_{out}}{A_L} \tag{4.27}
\]

where the relationship \( A_L : \eta_L \) is obtained from the hypsographic curve. As long as the flow exit from the lake is subcritical (see §4.7), \( \eta_L \) can be treated as an upstream internal boundary condition for the downstream river reach. In unusual conditions, a lake may have a natural sill at the outlet that acts as a weir, leading to supercritical flow. When this occurs, the outlet flow rate must be a strict function of the depth above the sill, i.e. a boundary condition on both \( Q \) and \( \eta \) through the \( h : \eta \) relation and the \( Q : h \) relation for the sill. The \( Q : h \) relationship will be specific to the lake and the sill geometry. Similarly, as long as the flow into a lake is subcritical, \( \eta_L \) can be treated as a downstream internal boundary condition on the upstream river reach. Supercritical flows into a natural lake are unusual, with perhaps the exception of a waterfall (discussed below). For a supercritical flow into a lake, \( \eta_L \) will have no effect.

Waterfalls, also known as “free overfalls” have supercritical conditions where the water goes over the brink, so the downstream depth and flow rate have no effect on the upstream condition. The general form of the equation is similar to eq. (4.17), with [45] recommending for rectangular channels of breadth \( B \)

\[
Q = 1.65 \sqrt{gB} (h_b)^{3/2} \tag{4.28}
\]

\footnote{An exception being the North American Great Lakes}
where \( h_b \) is the water depth at the brink of the waterfall.

As a caution, the equations for various flow structures under different conditions will exhibit step behavior if naively implemented in a river network model; that is, few of the hydraulic equations have been blended to ensure smooth transitions between regimes (e.g. a change from inlet to outlet control for a culvert). To avoid spurious oscillations and slow convergence, blended equations should be implemented for smooth transitions.

The structures in a river network that interrupt a clean solution of the Saint-Venant equations have similarities with nonlinear devices in an electric circuit (§2.4.5). There remains an open question as to whether the BCR approaches of VLSI can provide improved models for river networks, or whether the approaches used in hydraulics might inspire new ideas for BCR models.

### 4.12 Floodplains

Floodplains are a challenge for river network modeling. Strictly speaking, overbanking water generates a 2D (or 3D) flow that is no longer amenable to the Saint-Venant equations. Nevertheless, there is a long history of creating approximations of floodplain flow in 1D models [42]. The most common approach is to alter the resistance coefficient (e.g. Manning’s \( n \)) to represent the relatively high level of drag on the relatively shallow-depth flow through the floodplains [11, 34]. A principal drawback to this approach is in the geometric difficulty in identifying non-overlapping cross-sectional descriptions in a sinuous river. A complimentary approach is developing “off-channel” storage to represent the slow-moving flood waters. Although the modeling mechanics for off-channel storage are straightforward – requiring specification of a net lateral exchange between the river channel and floodplain – characterizing an empirical function for the lateral exchange is a non-trivial matter [43].

More recently, 1D-2D hybrid models have been developed to represent fluxes through the floodplain using a 2D depth-integrated version of the Navier-Stokes equations for hydrostatic flow (often known as the
shallow-water equations). Such models appear to be the most robust future for river network modeling as the exchange between river channel and floodplain requires only standard frictional representation of the flow across the landscape \[48, 26\].

4.13 Connecting with estuaries

The majority of the rivers around the world find their ultimate downstream point in the ocean. The ocean provides a challenging boundary condition for a river network model as the daily tidal oscillation can cause flow reversals quite far upstream from the river’s mouth. In many ecological definitions, an “estuarine” segment of a river is denoted as brackish water regions that have seen upstream salt transport. However, tidal influences that affect water surface elevation further upstream than the fresh/salt gradient and will play a role in river dynamics.

Because salinity affects the fluid density and allows development of stratified flow (i.e. denser saline water flowing under lighter fresher water), the Saint-Venant equations may provide poor flow predictions in an estuarine section of a river. However, upstream of an estuarine section the Saint-Venant equations are suitable when provided with an appropriate boundary condition for the downstream tidal elevation.

Where rivers debouch into the ocean through a complex maze of channels in a river delta (e.g. Fig. 1.6), the adequacy of the Saint-Venant equations is questionable. There are a myriad of problems in identifying channel connectivity and geometry, particularly as a delta may include many 2D flows through emergent vegetation. For the present, a river network modeler should seek to use a point upstream of a river delta as the downstream boundary condition of a model.

4.14 Summary

River networks form a complex web of physical processes affected by the interplay between water surface elevation, channel slope, and cross-sectional shape. The ratio of wave speed and water speed in the definition of flow criticality provides two very different possible flow regimes.
Subcritical flow, which dominates river networks, provides upstream propagation of forces even as the water moves downstream. Beyond the classic visualization of a river network as a web of channels, there exist both natural and man-made structures that punctuate the flow providing hydraulic controls. Identifying and representing these elements is a fundamental challenge for any river model.
Numerical Connections between Electric and River Networks

5.1 Introduction

Under the quasi-static assumption, an electric network can be modeled by using Kirchhoff’s laws, connecting lumped component models, or branch constituent relation (BCR), as presented in Chapter 2. Mathematically its overall behavior is described by a system of differential-algebraic equations (DAE’s). Using the hydrostatic and shallow water approximations, a river network can be modeled by the coupled 1D Saint-Venant equations, forming a system of distributed partial differential equations (PDE’s) as presented in Chapter 3, with sub-models for non-smooth physics as presented in Chapter 4. The obvious similarity is that both DAE’s and PDE’s can be formulated as nonlinear matrix systems, hence are solvable by similar numerical techniques; thus, the first step in crossing disciplines from electric circuit analysis to river network analysis is application of the numerical techniques developed in VLSI, which was done by [52] and is discussed below. Further possibilities for applying ideas for electric circuits are discussed in §6.

1 Equations and explanations in this chapter are substantially similar to portions of our work in [52], which was under preparation at the same time as this manuscript but is focused on the numerical method rather than the broader context of cross-
5.2 River channel PDE’s

From the range of PDE formulations that can be used to represent rivers (§3.7) we use the $Q$-$a$ formulation, presented in §5.2 below. Numerical solution for any system of equations requires choice of a discretization scheme (§5.3), application of boundary conditions (§5.4) and a choice numerical techniques (§5.5).

5.2 River channel PDE’s

A river network as divided into reaches between junctions, such that a reach is a 1D object. Junctions always occur at connections between tributaries, as well as where features (such as a dam, §4.11) create conditions where the Saint-Venant equations are inapplicable and auxiliary relationships are required (i.e. the equivalent of a BCR in an electric circuit). Thus, a reach (by definition) is a section of river that can be represented by a PDE. A reach is further subdivided into discrete segments for numerical solution.

The governing PDE’s for a reach are the continuity and momentum equations developed as eq. (3.41) and (3.45), repeated here for convenience:

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\beta Q}{a} \right) + ga \frac{\partial h}{\partial x} = ga (S_0 - S_f) \tag{5.2}
\]

where $Q$ is the flow rate, $a$ is the cross-sectional area, $h$ is the water depth, $\beta$ is the non-uniform momentum coefficient, $q_y$, $q_b$ are lateral and groundwater boundary flow rates (per unit length of river), $S_0$ is the bottom slope, and $S_f$ is the friction slope (§3.8), which is obtained from the Chezy-Manning equation, eq. (3.56), as

\[
S_f = \frac{1}{a} n^2 F Q^2 \tag{5.3}
\]

where the equivalent friction geometry $F$ is unique for each cross section specification, and is a function of the wetted area $a$ and perimeter $P_w$ as shown in eq. (3.55). The above are the classic 1D Saint-Venant
equations that can be discretized for a river network model. The variables $Q$ and $a$ are taken as the independent variables for prognostic (unsteady) solution, whereas $h(a)$ and $S_f(a, Q)$ are dependent variables with diagnostic equations. The friction coefficient, Manning’s $n$, and the momentum coefficient, $\beta$, are calibration variables that can be varied in each stream reach. However, $\beta = 1$ is usually everywhere applied and calibration is focused on $n$, which should be varied about accepted literature values as a function of the bed material (e.g. sand, gravel).

5.3 Discretization scheme

To numerically solve the Saint-Venant equations for each river reach, we first have to discretize the PDE functions of $f(a, Q)$. The literature in discretization methods is vast, but no single method has been proven superior. Here we use the Preissmann scheme, which is a non-staggered, finite difference, implicit four-point scheme with a long history in river modeling, dating back to [62]. In this scheme, the function $f$ is discretized as:

$$f(x, t) \simeq \frac{1}{2}(f_{j+1}^{n+1} + f_j^{n+1})$$  \hspace{1cm} (5.4)

where subscripts are the index for the discrete spatial node and superscripts are the discrete time level. Time and spatial derivatives are represented by:

$$\frac{\partial}{\partial x} f(x, t) \simeq \frac{1}{\Delta x} (f_{j+1}^{n+1} - f_j^{n+1})$$  \hspace{1cm} (5.5)

$$\frac{\partial}{\partial t} f(x, t) \simeq \frac{1}{2\Delta t} (f_{j+1}^{n+1} - f_{j+1}^{n} + f_j^{n+1} - f_j^{n})$$  \hspace{1cm} (5.6)

This biased implicit scheme is useful in river models as the flow is known \textit{a priori} to be in the $+x$ direction over most of the network, so the $n+1, j+1$ position is generally along the characteristic for flow from $n, j$.

Each reach is further subdivided into $N$ computational nodes with $N - 1$ computational segments connecting the nodes. The subdivision of reaches into smaller segments is required because the river auxiliary
5.3. Discretization scheme

relationships, i.e. \( h(a) \) and \( S_f(a, Q) \), can significantly differ over the length of a river reach. Junctions are represented by separate nodes from all connecting reaches through an internal boundary condition (§5.4). A convention is required for numbering segments and nodes; for clarity in exposition the computational segment \( j + 1/2 \) has an upstream node numbered \( j \) and a downstream node numbered \( j + 1 \). However, in a model code it is typically convenient to have integer indexes such that the \( j + 1/2 \) segment is actually given the index \( j \), the same as its upstream node.

With the above scheme, continuity eq. (5.1) is discretized on computational nodes as

\[
\begin{align*}
\begin{aligned}
\frac{a_{n+1}}{a_n + 1} + a_n &+ \frac{2\Delta t}{\Delta x}(Q_{n+1} - Q_n) \\
- \frac{\Delta t}{\Delta x}(q_{l})_{j+1/2} &+ \Delta t g \tilde{n}_{j+1/2}^2 \left( \left(Q_{n+1}^\prime \right)^2 F_{n+1} + \left(Q_{n+1}^\prime \right)^2 F_{j+1} \right) = 0
\end{aligned}
\end{align*}
\]  

(5.8)

Momentum eq. (5.2) is discretized as

\[
\begin{align*}
\begin{aligned}
\frac{Q_{n+1}}{Q_n} + Q_j &+ \frac{2\beta \Delta t}{\Delta x} \left[ \left(Q_{n+1}^\prime \right)^2 a_j - \left(Q_{n+1}^\prime \right)^2 a_j \right] \\
+ g \frac{\Delta t}{\Delta x} (a_{j+1} + a_j) (h_{j+1} - h_j) &- \Delta t g \left[ a_{n+1} + a_{j+1} \right] S_0(j+1/2) \\
+ \Delta t g \tilde{n}_{j+1/2}^2 \left( \left(Q_{n+1}^\prime \right)^2 F_{n+1} + \left(Q_{n+1}^\prime \right)^2 F_{j+1} \right) &+ \Delta t g \tilde{n}_{j+1/2}^2 \left( \left(Q_{n+1}^\prime \right)^2 F_{n+1} + \left(Q_{n+1}^\prime \right)^2 F_{j+1} \right) = 0
\end{aligned}
\end{align*}
\]  

(5.9)

Note that an unconventional \( \tilde{n} \) is used to represent Manning’s \( n \) from eq. (5.3) so as not to confuse the roughness coefficient with the time step, which is conventionally represented by superscript \( n \). The discrete value of \( \tilde{n} \) for a computational segment is given by the average of the
nodal values, i.e.

\[ \tilde{n}_{j+1/2} = \frac{1}{2} (\tilde{n}_j + \tilde{n}_{j+1}) \]  

which allows data to be provided uniformly at computational nodes, but appropriately distributed over computational elements.

### 5.4 Boundary conditions

A river network model has two types of boundary conditions: external and internal. The external conditions are the \( q_y \) and \( q_b \) for each computational element, along with \( Q \) (or possibly \( a \)) at the furthest upstream reach in the network and \( a \) at the furthest downstream point. External boundary conditions are readily implemented with standard numerical techniques. The major challenge for external boundary conditions in river network modeling is in coupling with a landscape hydrological model and groundwater model to obtain \( q_y \) and \( q_b \) throughout the network. Because of interactions between the river and the landscape, running a river network model in isolation is not useful, except as a theoretical exercise. The ubiquity of flow-type boundary conditions is a key difference between river and electric network modeling, in the latter the voltage (equivalent to water elevation) is the more common boundary condition (§4.2).

Internal boundary conditions are the conditions at the upstream and downstream end of each computational reach – i.e. at the junctions between the reaches defined in §5.3. For a river reach with \( N + 1 \) nodes, we have \( 2N + 2 \) unknowns to be solved (\( Q \) and \( a \) at each node). Because there are \( N \) segments, we will have \( N \) continuity equations, and \( N \) dynamic equations. These \( 2N \) equations plus two boundary conditions provide \( 2N + 2 \) equations, which is a properly defined system of nonlinear equations for a single reach.

For a river network with multiple river reaches, each reach is modeled by a set of Saint-Venant equations. To connect the Saint-Venant equations together, each tributary junction is represented by multiple “virtual” nodes, as shown in Fig. 5.1. Boundary conditions enforced at junctions are conservation of flow, eq. (4.15) and linear area functions,
5.4. Boundary conditions

Numerical implementation of internal boundary conditions depends whether subcritical or supercritical conditions occur at junctions (§4.7). Subcritical flow requires one upstream and one downstream boundary condition (either $Q$ or $a$ can be used). Reaches that are entirely supercritical require two upstream boundary conditions ($Q$ and $a$) and do not admit a downstream boundary condition. Reaches that transition from supercritical upstream to subcritical downstream require two boundary conditions upstream, one downstream, and a numerical scheme capable of handling the transcritical conditions that will occur within one of the computational segments [55, 66, 67]. Reaches with an upstream subcritical condition and a downstream supercritical condition will admit only a single upstream boundary condition and again will see transcritical conditions. Fortunately, the theoretical complexity for internal boundary conditions of different criticality is simplified in practice. With the exception of some mountain rivers and smaller streams in steep terrain, rivers are generally subcritical over long reaches. Furthermore, tributary junctions are inherently 3D flows that are poorly represented by 1D approximations (§4.10), so modeling junctions as subcritical is not unreasonable.

![Figure 5.1: Junction of three reaches that creates a virtual node; from [52].](image-url)
5.5 Numerical techniques

Although explicit time-marching techniques can be used to solve the Saint-Venant equations, the allowable time step for numerical stability will be small \[4, 81\]. An explicit model is limited by the Courant-Friedrichs-Lewy (CFL) condition for either flow velocity \((C_u)\) or for wave velocity \((C_w)\). As a result, the time step will be limited by the smaller of two conditions:

\[
\Delta t < \frac{C_u \Delta x}{Q/a} \quad (5.11)
\]

\[
\Delta t < \frac{C_w \Delta x}{\sqrt{gh}} \quad (5.12)
\]

where the values of \(C_u\) and \(C_w\) depend upon the particular numerical method, but typically \(C_u, C_w \leq 1\) is required for explicit methods. The wave velocity is often the limiting factor; for example, a river reach modeled with 1000 m between nodes with water depth of 3 m will require a time step less than 3 minutes for \(C_w < 1\), whereas typical river velocities on the order of 0.5 m/s would allow a time step of 1/2 hour using the same grid discretization for \(C_u < 1\). Some models use semi-implicit techniques \[65\], which apply implicit discretization of the free surface so that \(C_w > 1\) is allowed, along with explicit discretization of advection so that \(C_u\) controls the time step. Although semi-implicit schemes are well proven for subcritical ocean, lake and estuarine flows, they can be problematic on rivers as localized supercritical conditions (§4.7) occurring with high velocities require a small time step to satisfy the \(C_u\) condition. Although supercritical conditions are not common, a river network model must be stable in their presence because they will sometimes occur, particularly as transient phenomena (possibly as numerical artifacts) in a rapidly changing flow.

Fully-implicit time-marching methods allow large and stable time steps in a river network solution, which is a valuable attribute for unsteady flow conditions \[52, 76\]. When implicit methods are used to solve the Saint-Venant equations, at each time step a set of nonlinear equations, eq. (5.7) and eq. (5.9) must be solved. The most widely used method to solve systems of nonlinear equations is Newton’s method.
5.5. Numerical techniques

To solve a system of nonlinear equations \( \mathbf{f}(\mathbf{X}) = \mathbf{0} \), where \( \mathbf{X} \) is a vector of unknowns, and \( \mathbf{f}() \) is a system of nonlinear equations, Newton’s method requires an expansion of the nonlinear function around a starting point \( \mathbf{X}_0 \):

\[
\mathbf{f}(\mathbf{X}) \approx \mathbf{f}(\mathbf{X}_0) + \mathbf{J}(\mathbf{X}_0) \cdot (\mathbf{X} - \mathbf{X}_0)
\]  

(5.13)

Requiring the LHS of Eq. (5.13) to equal zero, we have:

\[
\mathbf{X} - \mathbf{X}_0 = -[\mathbf{J}(\mathbf{X}_0)]^{-1} \cdot \mathbf{f}(\mathbf{X}_0)
\]  

(5.14)

The Jacobian matrix of the nonlinear system \( \mathbf{J} \) consists of the derivatives of the nonlinear equations with respect to the unknown \( \mathbf{X} \), evaluated at the given starting point \( \mathbf{X}_0 \):

\[
\mathbf{J}(\mathbf{X}_0) = \frac{\partial \mathbf{f}(\mathbf{X})}{\partial \mathbf{X}} \bigg|_{\mathbf{X}=\mathbf{X}_0}
\]

\[
= \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \cdots \\
\frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots \\
\vdots & \vdots & \ddots \\
\end{bmatrix}
\bigg|_{\mathbf{X}=\mathbf{X}_0}
\]  

(5.15)

Hence the classic Newton’s method can be cast as an iterative process to search for the solution of the corresponding nonlinear equations from a starting point \( \mathbf{X}_0 \) as described in Algorithm 1.

Newton’s method is a local optimization method which finds the local minimum around the given starting point \( \mathbf{X}_0 \), which is not necessarily at the roots of the equation set. This issue is usually not a problem for time-marching solutions: the relative small change over \( \Delta t \) implies results from time \( n \) are a good starting position for \( \mathbf{X}_0 \) in solving for time \( n + 1 \). The classic version of Newton’s method (Algorithm 1) has a quadratic convergence rate in terms of the number of iterations required \([21]\), so it is computationally efficient.

To compute the Jacobian matrix of eq. (5.15), we can explicitly compute the derivatives of the continuity and dynamic equations shown
Algorithm 1 Classic Newton-Raphson Method

1: procedure NR\((t, m)\) \(\triangleright\) \(t\): given tolerance, \(m\): max number
2: \hspace{1em} iteration, \(k\): iteration counter
3: \hspace{1em} \(k \leftarrow 0\)
4: \hspace{1em} repeat
5: \hspace{2em} Evaluate \(f(X_k)\)
6: \hspace{2em} if \(\|f(X_k)\| < t\) then
7: \hspace{3em} return converged \(\triangleright\) Success
8: \hspace{2em} end if
9: \hspace{2em} Evaluate derivatives and populate Jacobian matrix \(J_k\)
10: \hspace{2em} Factorize Jacobian matrix \(J_k\)
11: \hspace{2em} Solve for corrections \(\Delta X \leftarrow J_k^{-1} \cdot f(X_k)\)
12: \hspace{2em} Update \(X_{k+1} \leftarrow X_k - \gamma \cdot \Delta X\)
13: \hspace{2em} \(k \leftarrow k + 1\)
14: \hspace{1em} until \(k > m\)
15: \hspace{1em} return nonconvergent \(\triangleright\) Nonconvergent within max iteration
16: end procedure

in eq. (5.7) and eq. (5.9). The derivatives of the continuity equation \(f_1()\) with respect to \(a^{n+1}\) are:

\[
\frac{\partial f_1}{\partial a_j} = 1.0 \quad (5.16)
\]

\[
\frac{\partial f_1}{\partial a_{j+1}} = 1.0 \quad (5.17)
\]

and the derivatives with respect to \(Q^{n+1}\) are

\[
\frac{\partial f_1}{\partial Q_j} = -\frac{2\Delta t}{\Delta x} \quad (5.18)
\]

\[
\frac{\partial f_1}{\partial Q_{j+1}} = \frac{2\Delta t}{\Delta x} \quad (5.19)
\]

Similarly the derivatives of the dynamic equation \(f_2()\) with respect
5.5. Numerical techniques

5.5.1. Numerical techniques

\[
\frac{\partial f_2}{\partial a_j} = \frac{2\beta \Delta t}{\Delta x} \left[ \frac{Q_j}{a_j} \right]^2 \\
- \frac{\Delta t}{\Delta x} \left[ a_j \frac{\partial h_{j+1}}{\partial a_j} + h_{j+1} - a_j \frac{\partial h_j}{\partial a_j} - h_j \right] \\
- \Delta t g S_0(j+1/2) + \Delta t g \bar{n}^2 \left[ Q_j \right]^2 \frac{\partial F_j}{\partial a_j}
\]

(5.20)

\[
\frac{\partial f_2}{\partial a_{j+1}} = - \frac{2\beta \Delta t}{\Delta x} \left[ \frac{Q_{j+1}}{a_{j+1}} \right]^2 \\
- \frac{\Delta t}{\Delta x} \left[ a_{j+1} \frac{\partial h_{j+1}}{\partial a_{j+1}} + h_{j+1} - a_{j+1} \frac{\partial h_j}{\partial a_{j+1}} - h_j \right] \\
- \Delta t g S_0 + \Delta t g \bar{n}^2 \left[ Q_{j+1} \right]^2 \frac{\partial F_{j+1}}{\partial a_{j+1}}
\]

(5.21)

and with respect to \(Q\)

\[
\frac{\partial f_2}{\partial Q_j} = 1 - 4\beta \frac{\Delta t}{\Delta x} \frac{Q_j}{a_j} + 2\Delta t g \bar{n}^2 Q_j F_j
\]

(5.22)

\[
\frac{\partial f_2}{\partial Q_{j+1}} = 1 + 4\beta \frac{\Delta t}{\Delta x} \frac{Q_{j+1}}{a_{j+1}} + 2\Delta t g \bar{n}^2 Q_{j+1} F_{j+1}
\]

(5.23)

Note that the above uses the equivalent friction geometry \(F\) at time \(n\). This approach effectively is a time-lagging of the friction term; i.e. neglecting the nonlinearity associated with the change of cross-sectional area and wetted perimeter from time \(n\) to \(n+1\) that is inherent in the \(F\) definition of eq. (3.55). This neglect of nonlinearity can lead to stability issues for large time steps or with rapidly changing friction [52].

Newton’s method requires the construction and factorization of the Jacobian matrix \(J\) at every iteration within a model time step. In general, the Jacobian matrix \(J\) can be any non-singular matrix; however, in most applications the Jacobian matrix is sparse. For the discrete system above, both continuity and momentum at any given node are functions only of the unknown flow rate and wetted area of two adjacent segments, \(Q_{j+1}, Q_j, a_{j+1} \) and \(a_j\), so the number of nonzero
Numerical Connections between Electric and River Networks

entries in each row of the Jacobian matrix is only four, as illustrated in Fig. 5.2. The exception being for nodes where internal boundary conditions are defined, which can have more nonzero entries depending on the type of node connection (e.g., a dam will have a different set of node connections than a tributary junction). However, since there are only a limited number of internal boundary conditions, their presence will not significantly change the sparsity of the Jacobian matrix. The sparsity of the Jacobian matrix has implications for the overall computational complexity. For a general, full, non-singular matrix of size $N$, the computational complexity of matrix factorization (or computing the matrix inverse) is $O(N^3)$. However, when the matrix is sparse, the complexity drops to $O(N^{1.2-1.3})$. For large river networks with tens of thousands to millions of computational nodes, the reduction of the complexity significantly reduces computational requirements.

![Typical sparsity structure of Jacobian matrix](image)

**Figure 5.2:** Typical sparsity structure of Jacobian matrix, from [52].

Theoretically, applying Newton’s method requires factorizing the Jacobian matrix every Newton iteration, with multiple iterations in each time step of a time-evolving solution for unsteady flow. Such matrix factorizations can be computationally expensive. However, the Jacobian matrix only needs to be constructed and factorized at each iteration to obtain quadratic convergence. If we can tolerate a slightly
5.5. Numerical techniques

non-optimal convergence rate, we can reduce the number of matrix factorizations in an unsteady flow solution. To explain this point, the correction step in the Newton’s method can be expressed as:

$$\Delta \mathbf{X} = -[\mathbf{J}]^{-1} \cdot \mathbf{f}(\mathbf{X}_0)$$  \hspace{1cm} (5.24)

One way to interpret the above equation is that \( \mathbf{J} \) (or more precisely, its inverse) provides the direction of the correction while the function value \( \mathbf{f}(\mathbf{X}_0) \) determines the magnitude; i.e. if \( \mathbf{f}(\mathbf{X}_0) \equiv 0 \), no correction will be applied, regardless of the Jacobian. The key point is that an imprecise Jacobian (i.e. one not updated at every iteration) will converge to the same solution of the nonlinear equations as a precise Jacobian, but simply requires more iterations. The theoretical convergence rate of this “Newton’s method with Jacobian by-pass” is linear (as contrasted to the quadratic convergence of the full Newton’s method). Despite increasing the number of iterations, the Jacobian by-pass approach can reduce the total computational expense when the cost of each iteration is reduced by more than the increase associated slowed convergence. Newton’s method with the Jacobian by-pass is shown in Algorithm 2.

To see how the Jacobian by-pass approach can help reduce computational costs, consider a typical numerical algebra package that solves for \( \mathbf{X} \) in a system of linear equations \( \mathbf{A}\mathbf{X} = \mathbf{B} \). Many linear algebra packages provide results in the form \( \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} \) where the factorized matrix \( \mathbf{A}^{-1} \) is stored in place, i.e., in the same memory block as the original matrix \( \mathbf{A} \). When a new right-hand-side \( \mathbf{B} \) is provided, the new \( \mathbf{X} \) solution is efficiently computed at a fraction of the CPU time used to factorize the original matrix \( \mathbf{A} \). Table 5.1 shows measured CPU time of matrix factorization and re-solving time for additional solutions for matrices at different sizes. The solution speed increases by a factor of 60\( \times \) for large matrices. Thus, a single Newton iteration of eq. (5.24) is significantly faster if Jacobian factorization is by-passed during the iteration; i.e. \( \mathbf{J}^{-1} \) is not recomputed, but is simply re-used from the prior iteration. The speed up will be smaller than 60\( \times \) because computing a new \( \mathbf{f}(\mathbf{X}) \) is still required at each iteration. This technique has been used for simulation problems in other disciplines, including circuit simulation [1].

The Jacobian by-pass makes a trade-off between the increased num-
Algorithm 2 Newton’s method with Jacobian by-pass

1: procedure NR_ACC(t, m, c) \(t\): tolerance, \(m\): max number iteration, \(c\): conditional factorize
2: \(k \leftarrow 0\)
3: flag \(\leftarrow 1\)
4: repeat
5: \(\text{Evaluate } f(X)\)
6: if \(\|f(X_k)\| < t\) then
7: \(\text{return converged}\) \(\triangleright \text{Success}\)
8: end if
9: if flag then \(\triangleright \text{Only factorize Jacobian when necessary}\)
10: \(\text{Evaluate derivatives and populate Jacobian matrix } J\)
11: \(\text{Factorize Jacobian matrix } J\)
12: flag \(\leftarrow 0\)
13: end if
14: \(\text{Solve for corrections } \Delta X \leftarrow J^{-1} \cdot f(X_k)\)
15: \(\text{Update } X_{k+1} \leftarrow X_k - \gamma \cdot \Delta X\)
16: \(k \leftarrow k + 1\)
17: if \(k \geq c\) then
18: \(\text{flag } \leftarrow 1\) \(\triangleright \text{Trigger Jacobian factorization}\)
19: end if
20: until \(k > m\)
21: \(\text{return nonconvergent}\) \(\triangleright \text{Nonconvergent within max iteration}\)
22: end procedure

For unsteady solution of river networks with the discretization methods outlined above, the Jacobian by-pass shows a net reduction of total computational time. Note that small network of only a few dozens of nodes will not see significant speed-up. Because computational time for factorization \((J^{-1})\) grows faster than computation time for \(f(X)\) with increasing matrix size, larger networks will generally see greater speed improvements.

There is a hidden danger in the Jacobian by-pass method. Because the Jacobian is not factorized for each iteration, the direction provided
Table 5.1: Measured runtime of various sparse matrix sizes and the corresponding factorization and solve time using UMFPACK 5.5. All matrices are sparse, asymmetric with stencil of five.

by the (stale) Jacobian for one or more computational nodes can be wrong. The incorrect direction can cause convergence problems, which can be detected during iteration and addressed by recalculating the Jacobian factorization and/or reverting to the full Newton’s method. Such conditions seldom occur in simulations of real-world river networks, but are more likely to occur when starting a network from a set of approximate initial conditions that might not reflect a physically realistic system. Algorithm 2 includes convergence checking for Jacobian factorization so that re-computation of the Jacobian occurs when convergence problems are detected.

It is important to emphasize that the Jacobian by-pass algorithm converges to the same solution of the original nonlinear equations as the full Newton’s method. This capability is different from methods that accelerate the solution by using some form of linearization to solve an approximation of the nonlinear equations, e.g. methods employing the “double-sweep” algorithm [17].

5.6 Smoothness of equivalent friction geometry

In Newton’s method (§5.5), the entries in the Jacobian matrix are derivatives of nonlinear functions, e.g. eqs. (5.16) – (5.23). To ensure
convergence, the derivatives within the Jacobian have to be continuous and smooth; i.e. the nonlinear functions have to be $C^1$ so both the function itself and its first-order derivatives are continuous [21]. This requirement creates a problem for nonlinear functions in the Saint-Venant equations as there is a direct dependence on cross-sectional area ($a$) in eqs. (5.1) and (5.2), as well potential issues where the friction slope ($S_f$) introduces the equivalent friction geometry, $F(a)$, and Manning’s $n$, eq. (5.3). In natural systems there is no reason to expect either the geometry or friction parameters to be $C^1$ smooth. Fig. 5.3(a) shows an example of non-smoothness in a typical surveyed river cross section. Such non-$C^1$ cross-sections provide non-$C^1$ derivatives that can cause convergence failure during Newton iteration. Numerically, this problem can be somewhat addressed in a solution algorithm by automatically reducing the time step when convergence fails. Where the non-smoothness is localized, the reduced time step may allow the algorithm to get close enough to the discontinuous solution such that the overall convergence tolerance is reached – i.e. the underlying problem is simply ignored if it is small enough. However, there is no way to algorithmically ensure that convergence can be reached for all possible forms of non-$C^1$ Jacobians. Thus, the presence of non-smooth functions causes increased computational time and reduces model robustness. This problem is better mitigated at its source – in the discontinuities of the natural data. As the Saint-Venant equations are PDE’s derived from continuum mechanics, it is not an unreasonable burden to require that their boundary conditions and associated geometrical definitions should be sufficiently smooth to satisfy $C^1$. A simple approach is to fit approximating cubic splines to the depth-area relation $h(a)$, a shown in Fig. 5.3(b), and $F(a)$ as shown in Fig. 5.3(c). Cubic-spline functions are, by definition, $C^1$ [18]. Furthermore, we can carefully select the “knots” in the construction of the spline function, so that a smooth characteristics are maintained despite the underlying non-smooth features of the natural system.
5.6. Smoothness of equivalent friction geometry

Figure 5.3: Surveyed data of typical cross-section from natural river segment in central Texas (a) raw $y-z$ survey data; (b) depth-area and derivative relationships (symbols), lines represent spline; (c) friction-depth and derivative relationships (symbols), lines represent spline. Data courtesy of City of Austin, after [52].
5.7 Numerical examples

As a first example, the effectiveness of Jacobian by-pass (§5.5) and the spline smoothing algorithm (§5.6) are demonstrated on two river reaches of Waller Creek, located in Central Texas, USA. The key characteristics as well as the simulation setup are listed in Table 5.2.

<table>
<thead>
<tr>
<th>Case</th>
<th>total river length (km)</th>
<th>simulated time (h)</th>
<th>time step (s)</th>
<th>number of nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>11.6</td>
<td>20</td>
<td>60</td>
<td>288</td>
</tr>
<tr>
<td>C2</td>
<td>6.4</td>
<td>35</td>
<td>60</td>
<td>149</td>
</tr>
</tbody>
</table>

Table 5.2: Model configuration for unsteady flow test simulations. “Simulated time” refers to the number of hours in the event simulated by the model.

The simulated results of the unsteady water surface at one computational node are shown in Fig. 5.4. The upper panel illustrates the results of using classic Newton’s method versus using Newton’s method with Jacobian bypass. As expected, there are no differences in the computed surface elevations. The bottom figure shows the difference when the original $y - z$ surveyed data were used versus sub-sampled and smoothed $h(a)$ and $F(a)$. Although there are minor differences, they are within acceptable range considering the uncertainty within surveyed cross section data. Indeed, it is questionable as to whether the solution of PDE’s with the raw non-$C^1$ data is, in any practical sense, “better” than the solution of PDE’s with smoothed $C^1$ data.

A computational time comparison of the four experimental scenarios are listed in Table 5.4. By applying Jacobian bypass (§5.5), the average number of Newton iterations is higher. However, the gain from the faster linear matrix solution more than compensates for the increased number of iterations. Also by using smoother, sub-sampled spline functions for $F(a)$ and $h(a)$, computational effort is further decreased as Newton’s method converges faster on a smoother problem.

A second example is a large river network located in Central Texas comprising the combined Guadalupe and San Antonio River basins. The river network includes 3,679 river reaches, with more than 1,800 junctions and 15,513 km of river channel length. A Saint-Venant model
5.7. Numerical examples

Figure 5.4: Water surface elevation (stage height) at a single node of a 6.4 km river reach modeled. (a) solutions with and without Jacobian bypass acceleration of the Newton-Raphson method. (b) solutions using raw \( y - z \) and smoothed equivalent friction geometry functions; from [52].
Numerical Connections between Electric and River Networks

<table>
<thead>
<tr>
<th>Case</th>
<th>Raw $h(A), F(A)$ w/o bypass</th>
<th>Raw $h(A), F(A)$ with bypass</th>
<th>Smooth $h(A), F(A)$ and \ w/o bypass</th>
<th>Smoothing $h(A), F(A)$ with bypass</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>12.1</td>
<td>17.0</td>
<td>13.1</td>
<td>11.9</td>
</tr>
<tr>
<td>C2</td>
<td>11.3</td>
<td>15.0</td>
<td>11.9</td>
<td>11.9</td>
</tr>
</tbody>
</table>

Table 5.3: Average Newton iterations per time step

<table>
<thead>
<tr>
<th>Case</th>
<th>Raw $h(A), F(A)$ w/o bypass</th>
<th>Raw $h(A), F(A)$ with bypass</th>
<th>Smooth $h(A), F(A)$ and \ w/o bypass</th>
<th>CPU time decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>8.4</td>
<td>4.5</td>
<td>3.4</td>
<td>60%</td>
</tr>
<tr>
<td>C2</td>
<td>4.1</td>
<td>2.3</td>
<td>1.6</td>
<td>61%</td>
</tr>
</tbody>
</table>

Table 5.4: Average CPU time (milliseconds) per time step. CPU time decrease is calculated from the raw data $h: F$ simulations without Jacobian bypass to the smoothed $h: F$ simulations with Jacobian bypass.

of this system was created with more than $1.4 \times 10^5$ computational nodes.

Simulation of a 14-day rainfall event required slightly more than 62 minutes of computer (CPU) time on a Linux desktop computer with 2.93Ghz Intel i7-870 CPU and 8GB of main memory, which means that the model computations were completed roughly $330 \times$ faster than the real event time. Snapshots of the simulated unsteady results are shown in Fig. 5.5

5.8 Summary

Although electric circuit simulation is based on solving systems of DAEs, the PDEs of river network can be discretized into a matrix form similar to those of DAEs. As with the circuit DAEs, the discrete PDE system is nonlinear and sparse, which allows the numerical solution techniques used in VLSI to be applied (e.g. Newton’s method with Jacobian by-pass). A key problem for the PDE system is that natural river geometries are not likely to be $C^1$ smooth, so convergence of Newton’s method, or indeed any method, can fail if appropriate methods for smoothing the geometry are not applied a priori.
Figure 5.5: Guadalupe and San Antonio Rivers. Color scale represents normalized channel cross-sectional area. Line width represents scaled flow rate; , after [52]
6

**Final Remarks**

### 6.1 A reconnection of disciplines

At the end of the 19th century, well-understood ideas in hydraulics were used to explain electric phenomena, as illustrated in Fig. 6.1 extracted from the first published volume of *The Electrical Journal*. Engineering students were generally exposed to hydraulics before electricity throughout the early 20th century, so hydraulics provided a common textbook metaphor for all engineers, e.g. pp. 9-10 in [27]. The pedagogical order has now been reversed: all engineering students see basic electric theory in high school and university introductory physics classes, while hydraulics is typically taught in the 2nd or 3rd year at university – and then only to civil and mechanical engineers. One might expect modern pedagogy for hydraulics to build on the electric foundations of their students, but apparently the connection between disciplines has been almost entirely forgotten or ignored.

The present reconnection of disciplines started with a meeting in 2009 between research scientists at the Center for Research in Water Resources at the University of Texas and IBM Research Austin. These two research centers have been only 2 km apart for more than 25 years, but were essentially oblivious to each other. During an in-
6.1. A reconnection of disciplines

Figure 6.1: Analogy of the Wheatstone bridge to water supply piping with a central valve from 1895 [9], digitized by GoogleBooks.

broductory meeting, the first author noted that a grand challenge for water modeling is the sheer computational expense of $10^6$ to $10^7$ nodes in a continental-scale river network [37]; the second author pointed out that such scales were routinely modeled in VLSI. Over the next several years, the authors learned how to communicate across the barriers of disciplinary jargon to see the commonalities that underlying electric circuits and river networks. The result was the Simulation Program for RIver NeTworks (SPRINT) that takes advantage of the numerical advances in VLSI that can be directly applied to river hydraulics [52, 51].

The research completed to date is only a starting point for cross-disciplinary transfer of ideas. The approach developed in §5 is an application of existing VLSI numerical methods to existing discretization methods in river hydraulics – essentially a one-way transfer. We believe there are much richer insights to be developed from the different approaches taken by disciplines.
6.2 Similarities and contrasts

These disciplines (and others) share a common modeling philosophy for complex computational problems: the more general (computationally expensive) physical laws are reduced and specialized (made computationally inexpensive) to focus on pertinent engineering problems. This pattern is seen at the boundaries between science and engineering throughout history: generalized models provide insights, while specialized models provide efficient answers. The analogies explored herein have developed because both electric theory and fluid mechanics evolved from conservation laws – charge and magnetic flux for the former, mass and momentum for the latter. For example, Gauss’ law can be written for either discipline as:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{F} = 0 \]  

(6.1)

where \((\rho, \mathbf{F})\) represents the fluid density and mass flux field in fluid mechanics or the charge density and charge flux field in electricity.

However, the similarities between rivers and electric circuits extend beyond the basic conservation laws. The key commonality is that both rivers and circuits take complex 3D physics and extract out dominant 1D behavior. At the smallest 3D scales, both disciplines have uncertainties in their knowledge of physical dimensions and nonlinear behavior that must be dealt with in up-scaling to practical models. We believe that these areas provide the greatest potential synergies for future advances.

Despite the fundamental similarities, the disciplines take substantially different approaches to reducing and specializing the general equations for engineering applications. In electric circuits, Maxwell’s equations are simplified to Kirchhoff’s laws and individual component models, or BCR’s, under the quasi-static assumption – which effectively reduces the dimensionality of the problem by decoupling inductive, capacitive, and resistive interactions. In fluid mechanics the 3D Navier-Stokes equations are simplified to 1D Saint-Venant equations, and then small-scale interactions (e.g. friction, non-hydrostatic pressure) are wrapped into empirical sub-models to reduce the number of variables.
6.2. Similarities and contrasts

The energy representation in the two disciplines are a further difference. In electric circuits, the KVL is equivalent to the concept that traversing any loops generate zero potential differences, which is effectively a statement of energy conservation. In a river network, friction converts mechanical into thermal energy, but the thermal energy has no dynamical effect on motion due to the large heat capacity of water; so energy (mechanical) is always lost from the system.

Different equation systems are produced by the two different approaches: electric circuit analyses has lumped models and DAE’s, whereas fluid mechanics has continuum mechanics PDE’s that distribute the coupled solution along the length of an element. In the lumped circuit models, the branch currents are uniquely determined by the node voltages and BCR’s for most devices. Hence during circuit analysis, there is no need to keep both branch currents and node voltages. For example, for a resistor with known resistance $R$, the relationship between its branch current and node voltages is uniquely determined by Ohm’s law. Therefore the “nodal” circuit analysis method is applied by simply skipping the branch current. For a few devices where the nodal analysis approach does not work, we can augment the equations with additional branch currents. This approach is used for independent voltage sources that are given node voltage drops, but have their branch currents determined by the rest of the circuit and KCL. Computationally this Modified Nodal Analysis (MNA) approach \cite{36} means that the number of unknowns in each circuit is only slightly larger than the number of nodes in the circuit. This approach does not have a direct analogue for river networks, where the distributed nature of the model requires two independent variables at each node (e.g. $Q$, $a$) that are nonlinearly related to auxiliary variables (e.g. $F$, $h$). Therefore the number of river network unknowns is at least twice the number of computational nodes.

At numerical simulation level, the circuit and river network models have strong similarities. Superficially both models present themselves as networks or, more precisely, networks of one-dimensional subcomponents. The network nature of these problems paves the way for efficient numerical simulation methods, principally due to the limited
connectivity of nodes. In the past, the complexity of PDE models for large river networks has driven many modelers towards explicit time-marching solution methods. However, the insight from electric circuit solutions (as illustrated in [52]) is that the matrix Jacobian for a river network PDE system can be treated similarly to the Jacobian for a circuit network DAE system, so implicit time-marching methods developed for VLSI can be readily used.

A key difference between a simple electric conductor and a river reach is that the river geometry (cross-sectional area, $a$) nonlinearly changes in concert with the flow rate ($Q$). So one must imagine an equivalent electric conductor whose cross-sectional area can expand or contract as a function of current (see §1.4). Furthermore the flow resistance ($S_f$) and potential energy gradients ($\partial \eta / \partial x$) in a river have a complex interplay with $Q$, so one must also imagine an equivalent conductor whose resistance and inductance have nonlinear feedback interactions with both current and geometry. Because of these complexities, river networks do not appear to have any simple parallel in the lumped approach for used for electric circuit analysis that allows inductance and capacitance to be separated from resistance in lumped linear devices. A simple conception of a river reach as a resistance device fails, as the change in depth ($h$) is a way in which a river locally stores potential energy that does not seem to be separable from the geometry and flow rate – inherently a river reach can embody coupled PDE properties similar to inductive, capacitive, and resistance electric devices; but for a river reach the relationships are inherently nonlinear and inseparable.

6.3 Future paths

The major difference between river and electric circuit networks is in the use of the lumped “nodal approach” in the latter, and there is an open question as to whether these ideas can ever be applied or adapted to river network modeling. In an electric circuit, energy is either in an electric field or a magnetic field and the transition between the two energy forms occurs at the speed of light; thus, the quasi-static lumped
circuit approach is neglecting dynamics at extremely small time scales. In contrast, the fluid energies in rivers are kinetic (velocity) and potential (water surface elevation), with the time for dynamic conversion scaling on the gravity wave speed, which is typically only one or two orders of magnitude faster than the fluid velocity and therefore cannot be neglected. Thus, the two energy forms are tracked separately and directly coupled in the distributed PDE model. There is an intriguing question as to whether or not there might be some entirely new way to re-think mass and momentum conservation to create something akin to the nodal approach. If some new formulation could be identified, a wide variety of VLSI modeling approaches predicated on the lumped model approach could be directly applicable to river networks.

There are a variety of other areas where electric circuit analyses may provide new advances for river modeling. The most likely areas for rapid progress are where electric circuit analysis has been able to make use of fast models, which have not been previously available for river networks. For example, in electric circuit analysis, small signal sensitivity computation is routinely performed to tune a circuit. Transferring these ideas to produce efficient sensitivity computation for river networks could provide new approaches for calibration, ensemble forecasting, and uncertainty quantification.

Electric circuit analyses might also provide new insights to river network modeling through event-driven simulation techniques. In circuit analysis, when certain parts of circuit are inactive a model can typically by-pass inactive sub-circuits without affecting the overall accuracy. Similarly, any large-scale river network is likely to have a majority of tributaries that are close to steady flow, while others might be undergoing rapid changes. Events can take hours to days for their signals to propagate through the river network, so numerical techniques that can handle internal latency and different time-rates of change in the network might prove effective.

There are also interesting questions as to how electric circuit analyses might benefit from insights developed in river network modeling. Although river network models have not developed the same level of numerical sophistication as seen in VLSI, they have had to deal with
issues of nonlinear coupling at multiple time scales that simply are not present in the lumped quasi-static approach. The river network experience in handling uncertainty and nonlinear coupling may be useful in VLSI as the size of semiconductor devices approaches atomic limits. It is becoming increasingly difficult to precisely make and model these tiny devices, resulting in manufacturing-induced “process variations” that are becoming a concern for VLSI design and analysis [7]. There are ongoing efforts to account for process variations in VLSI that might benefit from the approaches developed in hydrological modeling that separate behavior uncertainties due to uncertain data from the loss of fidelity due to model approximation.

The present work provides the foundations for future dialogue between electric and hydraulic engineers. In a sense, what has been accomplished to date is merely a harvest of the proverbial “low hanging fruit,” but we believe there are many possibilities for further transfer of ideas and methods between disciplines.
The authors would like to acknowledge: Dr. Peter Feldmann at IBM Research for the insightful discussions; Dr. Cedric David for providing data on the Guadalupe-San Antonio river basin; graduate students Fernando R. Salas and Prabhas R. Gupta for their work on the model simulations; and anonymous reviewers for their constructive suggestions. This manuscript was supported in part by the U.S. National Science Foundation CyberSEES grant CCF-1331768.
References


References


References


References


References


