NUMERICAL SIMULATION OF THE SHALLOW WATER EQUATIONS USING A
TIME-CENTERED SPLIT-IMPLICIT METHOD

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Abstract

The Shallow Water Equations (SWE) are widely used in the hydraulic science and engineering. The time dependency and nonlinearity causes difficulties in numerical simulation of the SWE. To address these problems, a Time-Centered Split-Implicit (TCSI) method is proposed to solve depth-averaged SWE. To verify the TCSI method, a progressive wave in an open channel is simulated. An artificial damping layer (or sponge layer) is applied in the outlet boundary. Results from the numerical experiments show the validity of TCSI in simulating depth-averaged SWE and the success of sponge layer in preventing the reflection from the boundary.

Introduction

The Shallow Water Equations (SWE) describe hydrostatic flow with a free surface. The depth-averaged (or two-dimensional) SWE are obtained by integrating the 3D incompressible Navier-Stokes equations over the water depth with the hydrostatic and Boussinesq approximations (Vreugdenhil 1994). The time dependency and nonlinearity of the SWE create difficulties in discretization (Bourchtein and Bourchtein 2006). Fully explicit methods such as explicit leapfrog (Cho and Yoon 1998; Fujima and Shigemura 2000), Runge-Kutta and MacCormack schemes (Fennema and Chaudhry 1990; Garcia and Kahawita 1986) are straightforward and easy to implement. However, they are restricted by the Courant-Friedrichs-Lewy (CFL) condition, which may require impractically small time steps. Appropriate numerical treatments are often used to stabilize these explicit schemes (Agoshkov et al. 1994; Peyret and Taylor 1983). Although fully implicit schemes may be unconditionally stable for larger time steps, the nonlinear problem must typically be posed as an approximate linear problem with an outer iteration (e.g. Picard or Newton) (Claudio and Mario 1994). Between explicit and implicit methods, semi-implicit methods have been developed and widely used. Casulli (1990) proposed a semi-implicit algorithm using Lagrangian treatment of nonlinear advection to computationally linearize the equation set. Although the Casulli (1990) approach has been successful and stable, its accuracy for CFL>1 is relatively poor (Hodges, 2004) In this paper, we use a Time-Centered Split-Implicit (TCSI) method proposed by Fu and Hodges (2007), to solve depth averaged SWE. In the TCSI method, the advection term is computationally linearized using a time-centered approximation without iteration or computation of additional time derivatives. The system is computationally split into two steps, both being linear matrix problems.
Governing Equations

The depth-averaged SWE are a system of nonlinear hyperbolic partial differential equations and written as:

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -g \frac{\partial \zeta}{\partial x} + \nu \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \tag{1}
\]

\[
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -g \frac{\partial \zeta}{\partial y} + \nu \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \tag{2}
\]

\[
\frac{\partial H}{\partial t} + \frac{\partial HU}{\partial x} + \frac{\partial HV}{\partial y} = 0 \tag{3}
\]

\[
\zeta = H + Z_b \tag{4}
\]

where $U$ and $V$ are depth-averaged velocities in $x$ and $y$ directions; $g$ is the gravitational acceleration; $H$ is the water depth and $\zeta$ is surface elevation; $Z_b$ is the bottom elevation. The above statement of the 2D SWE neglects the turbulence closure and bottom stress as a test case for our numerical method.

Numerical Model

Fu and Hodges (2007) proposed a Time-Centered Split-Implicit (TCSI) scheme to solve nonlinear advection equations. TCSI method is derived from the midpoint rule applied over the $n$ and $n+1$ time steps (in contrast to the application over three levels, $n-1$, $n$, and $n+1$ used in the leapfrog explicit method). A time-centered linear approximation for $n+1/2$ values is introduced into the nonlinear advection term. The discrete equation is split into an implicit solution for an intermediate variable (e.g. $U^*$) followed by an implicit solution for the $n+1$ values (e.g. $U^{n+1}$) as a function of the intermediate values. The approach provides two linear implicit equations whose sum is a second-order approximation of the original nonlinear equation.

Applying the TCSI method along with central-differencing of spatial derivatives on an Arakawa C grid (Arakawa and Lamb, 1977), we can write the implicit steps for solution of intermediate variables ($U^*, V^*, \zeta^*$) as

\[
V^* = V^n + \left[ -U^n \delta x V^n - V^n \delta y V^n + \nu \left( \delta_x^2 V^n + \delta_y^2 V^n \right) - g \delta_x \zeta^n \right] \frac{\Delta t}{2} + O(\Delta t)^3 \tag{5}
\]

\[
\zeta^* = \zeta^n - \left[ \delta_x \left( H^n U^n \right) + \delta_y \left( H^n V^n \right) \right] \frac{\Delta t}{2} \tag{6}
\]

\[
U^* = U^n + \left[ -U^n \delta x U^n - V^n \delta y U^n + \nu \left( \delta_x^2 U^n + \delta_y^2 U^n \right) - g \delta_x \zeta^* \right] \frac{\Delta t}{2} + O(\Delta t)^3 \tag{7}
\]
where $\zeta^* - Z_s = H^*$ and we use the shorthand notation $\delta_x$ and $\delta_y$ for spatial derivatives in $x$ and $y$. The implicit steps for the $n+1$ values are sequentially solved as

$$
U^{n+1} = U^* + \left[ -U^* \delta_x U^{n+1} - V^* \delta_y U^{n+1} + v \left( \delta_x^2 U^* + \delta_y^2 U^* \right) - g \delta_x \zeta^* \right] \frac{\Delta t}{2} + O(\Delta t)^3 \tag{8}
$$

$$
\zeta^{n+1} = \zeta^* + \left[ -\delta_x \left( H^* U^{n+1} \right) - \delta_y \left( H^* V^{n+1} \right) \right] \frac{\Delta t}{2} + O(\Delta t)^3 \tag{9}
$$

$$
V^{n+1} = V^* + \left[ -U^* \delta_x V^{n+1} - V^* \delta_y V^{n+1} + v \left( \delta_x^2 V^* + \delta_y^2 V^* \right) - g \delta_y \zeta^{n+1} \right] \frac{\Delta t}{2} + O(\Delta t)^3 \tag{10}
$$

Thus, the original set of coupled nonlinear equations becomes a sequence of linear implicit equations for each variable that is effectively a predictor-corrector approach. The split system is easy to solve as each step is a simple tridiagonal system.

**Results**

To verify our numerical method, a progressive wave in an open channel is simulated. The numerical model has been derived, implemented and run for two-dimensional flow; however, the test case presented here has only one-dimensional physics so that we can ensure the model does not show any spurious cross-dimensional effects. The progressive wave is introduced as an oscillating inlet boundary condition for an initially-quiescent channel with no through flow. Developing non-reflective outlet boundary conditions has been the subject of extensive research (e.g. Blayo and Debreu 2005; Tsynkov 1998), but for the present purposes as simple artificial damping layer (or sponge layer) is adequate. The sponge layer is applied upstream of the outlet boundary and downstream of the “test section” (i.e. the computational domain of interest) as an artificial damping function over a range of grid cells so as to dissipate the surface wave and its reflections before they propagates back into the test section (Durran 1999). The sponge layer damping function uses an increasing viscosity from the end of the test section to the outlet (Vinayan 2003). We use an inviscid progressive wave as a test case, so the viscosity is set to be zero within test section. The viscosity is represented as a function of position such that $0 \leq \nu(x) \leq \nu_{\text{max}}$ and

$$
\nu(x) = \nu_{\text{max}} f(x) \tag{11}
$$

If the artificial viscosity changes rapidly, spurious reflections may occur from the location where the viscosity first becomes non-zero. A gradual change, can be invoked by defining $f(x)$ as

$$
f(x) = \frac{1}{2} \left[ 1 - \cos \left( \pi \frac{x - x_r}{x_s - x_r} \right) \right] \tag{12}
$$

where $x_r$ and $x_s$ are the upstream and downstream limits of the sponge layer. Using the above definitions, a sponge layer is defined by its length $(x_s - x_r)$ and its maximum
viscosity $\nu_{\text{max}}$. In this test case, $x_r - x_s = 2\lambda$ where $\lambda$ is the wave length of the traveling wave; $\nu_{\text{max}}$ is selected by running test cases to determine a sufficiently large number that damps the wave and reflections. The sponge layer arrangement is shown in Figure 1.

\[
\nu \cdot \nabla \cdot \mathbf{u} = \lambda
\]

Figure 1. An open boundary rectangular channel with a sponge layer

Initially, the water is at rest with uniform depth,

\[
H(x, 0) = h_0
\]  \hspace{1cm} (13)

The inlet water level is forced by a cosine wave boundary condition

\[
H(0, t) = h_0 + a \cos(\sigma t) : t \geq 0
\]  \hspace{1cm} (14)

where $a$ is the wave amplitude and $\sigma$ is the frequency. This boundary condition generates a traveling wave with wave height $2a$. The time evolution of free surface at time $7T$ and $14T$ (where $T$ is the wave period) are shown in Figure 2. The wave shape is well-preserved well even after the 14th period. At time 7T, this wave has already reached the right boundary of the domain. The sponge layer prevents the reflection from the right boundary and the TCSI scheme appears to successfully simulate a progressive wave in an open boundary system.
Conclusions

A numerical model based on the TCSI method has been developed to simulate the depth averaged SWE. Simulation of a progressive wave in an open channel with an outlet sponge layer is tested in this paper. Our test results demonstrate the validity of the sponge layer boundary and the TCSI scheme. The TCSI scheme shows promise as an effective and efficient approach to obtaining a temporally second-order nonlinear method for advective equations.

References


